1. Splitting on Springer fibers

We have an equivalence of triangulated categories $D^b(D_{\lambda}(\tilde{\mathcal{D}})) \rightarrow D^b(\text{Mod}_\lambda U)$. Now we link them to the category of coherent sheaves on $X$.

We state the main theorem of this section. Recall $\mathcal{B}_{\lambda,\chi} = \mathcal{B}(1) \cap T^*_\lambda \mathcal{B}(1) \subseteq \tilde{T}^*_\mathcal{B}(1) \times_{\mathfrak{h}^*(1)} \{\lambda\}$.

**Theorem 1.1** ([BMR1]). For all integral $\lambda \in \mathfrak{h}^*$, the Azumaya algebra $\tilde{\mathcal{D}}$ splits on the formal neighborhood of $\mathcal{B}_{\lambda,\chi}$ in $T^*_\mathcal{B}(1) \times_{\mathfrak{h}^*(1)} \mathfrak{h}$.

As a consequence of Theorem 1.1, Morita theory gives equivalence of categories.

**Theorem 1.2.** We have equivalence of abelian categories

$$\text{Coh}_{\mathcal{B}_{\lambda,\chi}}(\tilde{T}^*_\mathcal{B} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*) \cong \text{Mod}_{\lambda,\chi} \tilde{\mathcal{D}};$$

$$\text{Coh}_{\mathcal{B}_{\lambda,\chi}}(T^*_\nu \mathcal{B}(1)) \cong \text{Mod}_\chi \mathcal{D}.$$

The rest of this section will be devoted to the proof of Theorem 1.1.

**Proposition 1.3** ([BG] § 3). Let $\chi = 0$, and $\zeta = (0, -\rho) \in \mathfrak{g}^{* (1)} \times_{\mathfrak{h}^*(1)/W} \mathfrak{h}^*_{\text{unr}}$, we have $U_0^{-\rho} \cong \text{End}_k(\delta^\zeta)$.

**Corollary 1.4.** Let $\mu^{(1)} : T^*_\mathcal{B}(1) \cong T^*_\mathcal{B}(1) \rightarrow N^*(1)$ be the moment map, then the natural map $\phi : \mu^{(1)}*U^{-\rho} \rightarrow \mathcal{D}^{-\rho}$ is an isomorphism.
Proof. The restriction of $\phi$ to the zero section $\mathcal{B}_{-\rho,0} \subseteq T^*\mathcal{B}(1)$ is an isomorphism, since up to a faithfully flat base change, every fiber of this map is the isomorphism $U^{-\rho} \to \mathcal{E}\text{nd}(\delta^\chi)$. Let $\mathcal{K}$ and $\mathcal{C}$ be respectively the kernel and cokernel of $\phi$. Then $\mathcal{C}$ restricted to $\mathcal{B}_{-\rho,0}$ is trivial, by the right exactness of restriction. Note that $\phi$ is $G$-equivariant, hence so are $\mathcal{C}$ and $\mathcal{K}$. Then by upper-semi-continuity, $\mathcal{C}$ is trivial, since every $G$-equivariant neighborhood of $\mathcal{B}_{-\rho,0}$ is the entire $T^*\mathcal{B}(1)$. Now we have a short exact sequence $0 \to \mathcal{K} \to \mu(1)^*U^{-\rho} \to \mathcal{D}^{-\rho} \to 0$, with $\mathcal{D}^{-\rho}$ locally free, restriction of this sequence to $\mathcal{B}_{-\rho,0}$ is exact. \(\square\)

Lemma 1.5. Let $U\widetilde{\rho}$ be the completion of $U$ at the Harish-Chandra central character $-\rho$. It is an Azumaya algebra over $\mathfrak{g}_{\mathcal{N}(1)}^\ast$, the formal neighborhood of $\mathcal{N}(1)$ in $\mathfrak{g}_{\mathcal{B}}^\ast$.

Proof. Note that $U\widetilde{\rho}|_{\mathcal{N}(1)} \cong U^{-\rho}$ is a matrix algebra. Only need to show that $U\widetilde{\rho}$ is locally free, which in turn amounts to show it is flat.

There are two facts: $\mathfrak{g}^\ast$ is flat over $\mathfrak{h}^\ast/W$; and $\mathfrak{u}(\mathfrak{g})$ is flat over $\mathfrak{h}^\ast/W$ for $p$ large enough. Therefore, $U\hat{0}$ is flat over $\mathfrak{g}_{\mathcal{N}(1)}^\ast$. So is $U\widetilde{\rho}$ which is a translation of $U\hat{0}$. \(\square\)

Corollary 1.6. For any closed point $\chi \in \mathcal{N}(1)$, $U\widetilde{\rho}$ is an Azumaya algebra on $\mathfrak{g}_{\chi}^\ast$, the formal neighborhood of $\chi$ in $\mathfrak{g}(1)$. Moreover, it splits on $\mathfrak{g}_{\chi}^\ast$.

To summarize, $\mathcal{D}$ splits on the formal neighborhood of $\mathcal{B}_{-\rho,\chi}$ in $\mathcal{T}^*\mathcal{B} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*$. Now we look at the effect of twisting by a group character on twisted differential operators. Let $\pi: \tilde{X} \to X$ the the torus torsor. We look at $(\pi_*\mathcal{D}_X \otimes_k k_\eta)^H$. This sheaf clearly has an action by $\mathcal{D}_X$. But this sheaf can also be interpreted as the isotypical component in $\pi_*\mathcal{D}_X$ transforms under $H$ by the character $\eta$. On the other hand, let $\tau_\eta$ be the translation automorphism on $\mathcal{T}^*\chi^0 \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*$ shifting the second factor by $\eta$. Then $\tau_\eta^*\mathcal{D}_X$ also acts on $(\pi_*\mathcal{D}_X \otimes_k k_\eta)^H$. One can check this bimodule induces Morita equivalence between $\tau_\eta^*\mathcal{D}_X$ and $\mathcal{D}_X$.

If $\mathcal{D}$ splits on the formal neighborhood of $\mathcal{B}_{-\rho,\chi}$, it also splits on the formal neighborhood of $\mathcal{B}_{\lambda,\chi}$ for integral $\eta$. This completes the proof of Theorem 1.1.

2. Affine braid group action

2.1. Review of affine braid group. For $\alpha$ a coroot and $n \in \mathbb{Z}$, let the hyperplanes $H_{\bar{\alpha},n}$ given by $\{\lambda \in \Lambda \mid \langle \bar{\alpha}, \lambda + \rho \rangle = np\}$. Open facets are called alcoves and codimension one facets are called faces. There is a special alcove, called the fundamental alcove, denoted by $A_0$, i.e., the alcove containing $(\epsilon + 1)\rho$ for small $\epsilon > 0$. It consists of those weights $\lambda$ such that $0 < \langle \lambda + \rho, \bar{\alpha} \rangle < p$ for all $\alpha \in \Phi^+$. The set of faces of $A_0$ will be denoted by $I_{aff}$.

Let $W_{aff} := W \ltimes Q$ be the affine Weyl group. It acts naturally on $\Lambda$ via the dot-action as follows. Elements in $W$ acts via the usual dot-action. Element $\nu$ in the lattice acts by $\lambda \mapsto \lambda + p\nu$. The group $W_{aff}$ is generated by reflections in
affine hyperplanes $H_{a,n}$. The $(W_{aff}, \bullet)$-orbits in the set of faces are canonically identified with $I_{aff}$, the faces in the closure of the fundamental alcove $A_0$. The (Coxeter) generators of the group $W_{aff}$ can be chosen to be the reflections in the faces of the alcove $A_0$.

For $\alpha \in I_{aff}$, let $s_\alpha \in W_{aff}$ be the reflection. Associated to $\alpha$ a standard generator $\tilde{s}_\alpha \in B_{aff}$. Then we define a set theoretical lifting $C : W_{aff} \to B_{aff}$, sending a minimal length decomposition $w = s_{\alpha_1} \cdots s_{\alpha_l(w)}$ to $\tilde{w} = \tilde{s}_{\alpha_1} \cdots \tilde{s}_{\alpha_l(w)}$. Then $B_{aff}$ can be presented as follows. The generators are taken to be the image of $C$, and relations are given by $\tilde{w}u = \tilde{w}\tilde{u}$ when $l(wu) = l(w) + l(u)$.

Similarly, the extended affine Weyl group $W_{aff}':= W \ltimes \Lambda$ has the length function extending that on $W_{aff}$. We write $W_{aff}'$ as $W_{aff} \ltimes \text{Stab}_{W_{aff}'}(A_0)$. Then the length function on $W_{aff}'$ is given by $l(w\omega) = l(w)$ for $\omega \in \text{Stab}_{W_{aff}'}(A_0)$. The extended affine Braid group $B_{aff}'$ can be presented in a fashion similar to the non-extended one. The generators are $\tilde{w}$ for $w \in W_{aff}'$, and relations are given by $\tilde{w}u = \tilde{w}\tilde{u}$ when $l(wu) = l(w) + l(u)$. As $\text{Stab}_{W_{aff}'}(A_0)$ permutes $I_{aff}$, we have naturally $B_{aff}' = B_{aff} \ltimes \text{Stab}_{W_{aff}'}(A_0)$. A smaller set of generators of $B_{aff}'$ can be chosen to be $I_{aff}$ and $\text{Stab}_{W_{aff}'}(A_0)$.

2.2. Review of intertwining functors. Note that $\text{Mod}_\lambda U = \text{Mod}_\mu U$ for any $\lambda$ and $\mu$ in the same $W_{aff}, \bullet$-orbit. For any $\lambda$, $\mu \in \Lambda$ we define $I_{\mu\lambda} : D^b(\text{Mod}_\lambda U) \to D^b(\text{Mod}_\mu U)$ as the composition $R\Gamma_{\tilde{\mathfrak{g}}_{\mu\lambda}} \circ \left(\mathcal{C}_{\mu - \lambda} \otimes \mathcal{C}_{\mathfrak{z}}^{-} \right) \circ \mathcal{L}^\lambda$. In the case when $\lambda$ and $\mu$ are in the same $W_{aff}, \bullet$-orbit and are both regular, this functor become an auto-equivalence.

The main goal of this section is to explain how these functors fit together to an affine braid group action. In characteristic zero, we have a braid group action on $D^b(\text{Mod}_\lambda U)$ for regular $\lambda$. (See e.g., [B] and [T].) The action of generators are built up using translation functors.

For $\lambda$, $\mu \in \Lambda$, we define $T^\mu_\lambda : \text{Mod}_\lambda U \to \text{Mod}_\mu U$ sending $M$ to $[V_{\mu - \lambda} \otimes M]_\mu$ here $V_{\mu - \lambda}$ is a finite dimensional representation with extremal weight $\mu - \lambda$, and $[-]_\mu$ means taking the component supported on the point $\mu$ in $h^*/W$. As this functor is exact, it has clear counterpart on the level of $D$-modules. On $\mathfrak{z}$ we take $\nu_{\mathfrak{g}}$ as the vector bundle corresponding to the $G$-module $V_{\mathfrak{g}}$. We have

$$T^\mu_\lambda(R\Gamma_{\tilde{\mathfrak{g}}_{\lambda}}M) = [V_{\mu - \lambda} \otimes R\Gamma_{\tilde{\mathfrak{g}}_{\lambda}}M]_\mu = [R\Gamma_{\tilde{\mathfrak{g}}_{\lambda}}(V_{\mu - \lambda} \otimes M)]_\mu \cong R\Gamma_{\tilde{\mathfrak{g}}_{\mu}}([V_{\mu - \lambda} \otimes M])_\mu.$$ 

The bundle $\nu_{\mathfrak{g}}$ has a filtration by line bundles, or better by $V_{\mathfrak{g}}[\nu] \otimes \mathcal{O}_\nu$ and the smaller $\nu$ appears earlier in the filtration.

**Proposition 2.1.**

1. If $\mu$ is in the closure of the facet of $\lambda$ ($\lambda \to \mu$ for short), then $T^\mu_\lambda(R\Gamma_{\tilde{\mathfrak{g}}_{\lambda}}M) \cong R\Gamma_{\tilde{\mathfrak{g}}_{\mu}}(\mathcal{O}_{\mu - \lambda} \otimes M)$.

2. If $\mu$ is regular and $\lambda$ lies in a codimension 1 wall $H$, and $s_H(\mu) < \mu$, then there is an exact triangle

$$R\Gamma_{\tilde{\mathfrak{g}},s_H}(\mathcal{O}_{\mu - \lambda} \otimes M) \to T^\mu_\lambda(R\Gamma_{\tilde{\mathfrak{g}}_{\lambda}}M) \to R\Gamma_{\tilde{\mathfrak{g}},\mu}(\mathcal{O}_{\mu - \lambda} \otimes M) \to [1].$$
To prove this proposition, we only need to count the weights occur in \((\lambda + \text{weights in } V_{\mu}) \cap W_{\mu}\). In case (1) there is only one which is \(\mu\). In case (2) there are two of them \(\mu\) and \(s_{H}\mu\), and \(\mu\) occurs later.

**Proposition 2.2.** If \(\nu \rightarrow \mu \rightarrow \lambda\), then \(T_{\mu}^{\nu} \circ T_{\lambda}^{\mu} \cong T_{\lambda}^{\nu}\) and \(T_{\nu}^{\mu} \circ T_{\lambda}^{\nu} \cong T_{\lambda}^{\nu}\). In particular, if \(\mu \rightarrow \nu \rightarrow \mu\) then \(T_{\mu}^{\nu} \cong T_{\mu}^{\nu-1}\).

**Proof.** By adjointness, we only need to prove one of them.

On the \(D\)-module level, twisting by line bundles composes as they should. This means \(T_{\mu}^{\nu} \circ T_{\lambda}^{\mu} \circ T_{\lambda}^{\nu} \cong T_{\lambda}^{\nu} \circ T_{\lambda}^{\mu} \circ T_{\lambda}^{\nu}\). Composing \(L\), and using the commutative diagram

\[
\begin{array}{ccc}
D^b(\text{Coh}_{\hat{S}_{\lambda}}) & \xleftarrow{L_{\lambda}} & D^b(\text{Mod}_{\lambda} U) \\
\text{Res}_{\hat{S}_{\mu}}^{\hat{S}_{\lambda}} & \downarrow & \text{Ind}_{\text{Mod}_{\lambda} U}^{\text{Mod}_{\lambda} \hat{U}} \\
D^b(\text{Mod}_{\mu} U) & \xleftarrow{L_{\mu}} & D^b(\text{Mod}_{\lambda} \hat{U})
\end{array}
\]

we have \(T_{\mu}^{\nu} \circ T_{\lambda}^{\mu} \circ \text{Res}_{\hat{S}_{\mu}}^{\hat{S}_{\lambda}} \text{Ind}_{\text{Mod}_{\lambda} U}^{\text{Mod}_{\lambda} \hat{U}} \cong T_{\lambda}^{\nu} \circ \text{Res}_{\hat{S}_{\mu}}^{\hat{S}_{\lambda}} \text{Ind}_{\text{Mod}_{\lambda} U}^{\text{Mod}_{\lambda} \hat{U}}\).

Then \(T_{\mu}^{\nu} \circ T_{\lambda}^{\mu}\) sits in the left hand side as a direct summand and \(T_{\nu}^{\mu}\) is a factor of the right hand side. We get \(T_{\mu}^{\nu} \circ T_{\lambda}^{\mu} \rightarrow T_{\nu}^{\mu}\). Applying them to the generator of the category \(U_{\lambda}\) to see that they are isomorphic as functors.

Now we have the translation functors we can use them to build the reflection functors and intertwining functors as in [B]. Assume \(\nu\) lies in a codimension 1 wall of the facet of \(\mu\). Define

\[R_{\mu|\nu} := T_{\mu}^{\nu} T_{\mu}^{\nu} : \text{Mod}_{\mu} U \rightarrow \text{Mod}_{\mu} U.\]

**Corollary 2.3.** If \(\mu\) is regular and \(\nu\), \(\nu'\) lie in the same codimension 1 wall, then \(R_{\mu|\nu} \cong R_{\mu|\nu'}\).

This means \(R_{\mu|\nu}\) depends only on the wall, not the character itself. As \(R_{\mu|\nu}\) is self-adjoint, we have two adjunctions. We define

\[\Theta_{\mu|\nu} := \text{cone}(\text{id} \rightarrow R_{\mu|\nu})\] and \(\Theta'_{\mu|\nu} := \text{cone}(R_{\mu|\nu} \rightarrow \text{id}).\)

**Corollary 2.4.** If \(\mu\) is regular and \(\nu\), \(\nu'\) lie in the same codimension 1 wall, then \(\Theta_{\mu|\nu} \cong \Theta'_{\mu|\nu}\).

In the case when \(\mu\) is regular, these two functors can be expressed as the intertwining functors defined as follows.

**Lemma 2.5.** When \(\mu\) is regular and \(\nu\) lies in a codimension 1 wall \(H\) in the facet of \(\mu\), and \(s_{H}\mu < \mu\), we have \(\Theta_{\mu|\nu} \cong I_{(s_{H}\mu)\mu}\) and \(\Theta_{\mu|\nu} \cong I_{\mu(s_{H}\mu)}\).

Note that \(\text{Mod}_{s_{H}\mu} U = \text{Mod}_{\mu} U\).
Proof. For any module $M$ we take the $\Gamma$-acyclic resolution of $\mathcal{L} M$. For acyclic $C$, have have from Proposition 2.1 (1) that $C \otimes \mathcal{O}(\nu - \mu)$ is acyclic. So is $[C \otimes \mathcal{O}(\mu - \nu) \otimes V_{\mu-\nu}]_\mu$. Using Proposition 2.1 (2) we know

$$C \otimes \mathcal{O}(s H \mu - \mu) \to [C \otimes \mathcal{O}(\mu - \nu) \otimes V_{\mu-\nu}]_\mu \to C \to [1],$$

hence applying $\Gamma$ we are done. \hfill \Box

2.3. The affine braid group action on representation categories. Now we describe how the functors $\Theta$ fit together to give an affine braid group action. As noted above, for any regular $\mu$, and an arbitrary $\nu$ lying in a face of the alcove containing $\mu$, the functor $\Theta \mid_{\nu}$ depends only on the wall containing $\nu$. For a regular $\lambda$, the faces of the alcove containing $\lambda$ are naturally labeled by $I_{aff}^\prime$. For regular $\lambda$, the orbit $W_{aff}^\prime \cdot \lambda$ is a free orbit. We define a right action of $W_{aff}^\prime$ on this orbit by $(u \cdot \lambda) w = uw \cdot \lambda$ for $u$ and $w \in W_{aff}^\prime$. For $w \in W_{aff}^\prime$ and $\mu \in W_{aff}^\prime \cdot \lambda$, we say $w$ increases $\mu$ if $\sigma_{s_1} \cdots \sigma_{s_i} < \sigma_{s_1} \cdots \sigma_{s_{i+1}}$ for all $i$, where $w = s_1 \cdots s_{l(w)} \omega$ is a reduced decomposition with $l(\omega) = 0$.

Lemma 2.6. Assume $\alpha \in I_{aff}$ and $\mu \in W_{aff}^\prime \cdot \lambda$ is such that $\mu s_\alpha > \mu$. Let $\mu w = \nu$ then

$$D^b(\text{Mod}_\mu U) \xrightarrow{\Theta \mid_{\nu}} D^b(\text{Mod}_\nu U),$$

where $\nu$ is in the face of the alcove containing $\mu$ labeled by $\alpha$.

Theorem 2.7 ([BMR2]). Let $\lambda \in \Lambda$ be regular. The assignment

$$\alpha \in I_{aff} \mapsto \Theta \mid_{\nu} =: \Theta_\alpha$$

for an arbitrary $\nu$ in the face of the alcove containing $\lambda$ labeled by $\alpha \in I_{aff}$, and

$$\omega \in \text{Stab}_{W_{aff}^\prime}^\prime (A_0) \mapsto T_{\omega}^\lambda =: T_\omega$$

defines a (weak) right action of $B_{aff}^\prime$ on $D^b(\text{Mod}_\lambda U)$.

The proof is the same as in [T].

Proof of Theorem 2.7. For $w \in \lambda W_{aff}^\prime$, let $w = \omega s_{\alpha_1} \cdots s_{\alpha_{l(\omega)}}$ be a decomposition with $l(\omega) = 0$ and $\alpha_i \in I_{aff}$. We have

$$(\mathcal{O}(\nu)_{\lambda \omega_{\alpha_1} \cdots \alpha_{l(\omega)} - \lambda} \otimes \mathcal{O}(\omega)) \circ \mathcal{L}^\lambda_{\omega_{\alpha_1} \cdots \alpha_{l(\omega)}} \cong \mathcal{L}^\lambda T_{\omega} \Theta_{\alpha_1} \circ \cdots \circ \Theta_{\alpha_{l(\omega)}}.$$

$\Box$
Proposition 2.8. Assume $w \in W'_\text{aff}$, and $\mu \in W'_\text{aff} \bullet \lambda$ is such that $w$ increases $\mu$. Let $\mu w = \nu$ then

$$D^b(\text{Coh}_{\mu} \tilde{\mathcal{D}}_{\mathcal{P}}) \xrightarrow{\ell^\nu - \mu \otimes \tilde{\mathcal{D}}_{\mathcal{P}}} D^b(\text{Coh}_{\nu} \tilde{\mathcal{D}}_{\mathcal{P}})$$

$$D^b(\text{Mod}_{\mu} U) \xrightarrow{\Theta^\Phi} D^b(\text{Mod}_{\nu} U).$$

2.4. Action on the level of $D$-modules. The followings are straightforward consequences of the construction of the affine braid group action.

Corollary 2.9. Fix a regular $\lambda \in \Lambda$. For $\nu \in \Lambda^+ \subseteq W'_\text{aff}$, let $\mu = \lambda + p\nu$. Then we have

$$D^b(\text{Coh}_{\lambda} \tilde{\mathcal{D}}_{\mathcal{P}}) \xrightarrow{\ell^\nu(\nu) \otimes \ell^\nu(1)} D^b(\text{Coh}_{\mu} \tilde{\mathcal{D}}_{\mathcal{P}})$$

$$D^b(\text{Mod}_{\lambda} U) \xrightarrow{\Theta^\Phi} D^b(\text{Mod}_{\mu} U).$$

Corollary 2.10. Fix a regular $\lambda \in \Lambda$. For $\nu \in \Lambda^+ \subseteq W'_\text{aff}$, we have

$$D^b(\text{Coh}_{\lambda} \tilde{\mathcal{D}}_{\mathcal{P}}(1) \tilde{\mathcal{G}}(1)) \xrightarrow{\ell^\nu(\nu) \otimes \ell^\nu(1)} D^b(\text{Coh}_{\mu} \tilde{\mathcal{D}}_{\mathcal{P}}(1) \tilde{\mathcal{G}}(1))$$

$$D^b(\text{Mod}_{\lambda} U) \xrightarrow{\Theta^\Phi} D^b(\text{Mod}_{\mu} U).$$

3. Translation functors on the level of coherent sheaves

As we need to consider the singular character $\lambda$, hence in order to do this we need to consider a singular version of the localization theorem. This is essentially the same as the regular case, except that $D^b(\text{Mod}_{\lambda, \chi} U)$ is localized to twisted $D$-modules on a partial flag variety.

Let $P \subseteq G$ be a parabolic subgroup, with unipotent radical $J$ and Levi $\tilde{P} = P/J$. Let $\mathcal{P} = G/P$ and $\tilde{\mathcal{P}} = G/J$ which is a $\tilde{P}$-torsor over $\mathcal{P}$. Let $\tilde{\mathcal{D}}_{\mathcal{P}}$ be $(\pi_* \mathcal{D})^{\tilde{P}}$. The sheaf of enveloping algebras is $\tilde{\mathcal{D}}_{\mathcal{P}}$. The total space of $\tilde{\mathcal{T}}$ is denoted by $\tilde{T}^{\mathcal{P}}$. Let $\tilde{\mathcal{G}}^{\mathcal{P}}$ be the subset of $\mathcal{P} \times \mathfrak{g}^*$ consisting of pairs $(\mathfrak{p}, \chi)$ with $\mathfrak{p} \in \mathcal{P}$ and $\chi \in \mathfrak{g}^*$ such that $\chi_{|\text{nilp}(\mathfrak{p})} = 0$. It is endowed with two projections

$p_{\mathfrak{g}} : \tilde{\mathcal{G}}^{\mathcal{P}} \to \mathfrak{g}^*$ and $p_{\mathfrak{p}^*} : \tilde{\mathcal{G}}^{\mathcal{P}} \to \mathfrak{p}^* =: \text{Lie}(\tilde{P})^*$. The center of $\tilde{\mathcal{D}}_{\mathcal{P}} =: \mathfrak{z}(\tilde{\mathcal{D}}_{\mathcal{P}}) \cong \ell^\nu(\nu) \otimes \ell^\nu(1)\tilde{\mathfrak{p}}$.

Note that there is a natural map $\tilde{\pi}^\mathcal{P} : \tilde{\mathcal{G}}^{\mathcal{P}} \to \tilde{\mathcal{G}}_{\mathcal{P}}$ such that $p_{\mathfrak{g}} = \tilde{\pi}^\mathcal{P} : \tilde{\mathcal{G}}_{\mathcal{P}} \to \mathfrak{g}^*$ factors through $\tilde{\pi}^\mathcal{P}$. As $\text{pr}_1$ is a proper morphism, so are $\tilde{\pi}^\mathcal{P}$ and $p_{\mathfrak{g}}$.

Let $\mathcal{P} = G/P$ be a partial flag variety. We say $\lambda$ is $\mathcal{P}$-regular if it has singularity exactly $\mathcal{P}$.
**Theorem 3.1** ([BMR2]). Under the assumption that $\lambda$ is $P$-regular, we have an equivalence of categories

$$R\Gamma_{\tilde{\mathcal{G}}_P,\lambda} : D^b(\text{Coh}_{\lambda,X} \tilde{\mathcal{G}}_P) \to D^b(\text{Mod}_{\lambda,X} U).$$

Similarly we have the notion of generalized Springer fibers and $\tilde{\mathcal{G}}_P$ splits on their formal neighborhoods. We summarize the equivalences of categories as follows

$$D^b \text{Coh}_{Z(\tilde{\mathcal{G}}_P) \times Z(U)(\chi,W_{\lambda}(\tilde{\mathcal{G}}_P)).}$$

We say an integral weight $\lambda \in \Lambda$ is $P$-unramified for a parabolic subgroup $P$, if the map $h^*/W_P \to h^*/W$ is unramified at $W_P \cdot \lambda$.

For two parabolic subgroups $P \subseteq Q \subseteq G$ and $\pi : \mathcal{P} \to \mathcal{Q}$. The natural map $\tilde{\pi}_P^* : \tilde{\mathcal{G}}_P \to \tilde{\mathcal{G}}_Q$ is also a proper morphism.

**Proposition 3.2.** [BMR2] For $P \subseteq Q \subseteq G$ be two parabolic subgroups, and $\mu$, $\nu \in \Lambda$ which are respectively $\mathcal{P}$ and $\mathcal{Q}$-regular unramified, for any $\chi \in g^*(1)$ we have

$$T_{\mu}^\nu \circ \gamma_{X,\mu} \cong \gamma_{X,\nu} \circ R\tilde{\pi}_{\mathcal{P}}^{(1)}$$ and $T_{\mu}^\nu \circ \gamma_{X,\nu} \cong \gamma_{X,\mu} \circ L\tilde{\pi}_{\mathcal{P}}^{(1)}.$

**Proof.** Again by adjointness, we only need to prove one.

$$T_{\mu}^\nu [R\Gamma(M_{X,\mu} \otimes \theta_{Z\tilde{\mathcal{G}}_P})] \cong T_{\mu}^\nu [R\Gamma(\tilde{\pi}_{\mathcal{P}}^{(1)*}(M_{X,\mu} \otimes \theta_{Z\tilde{\mathcal{G}}_P}))].$$

Let $\alpha$ be a positive root and $\mathcal{P} = \mathcal{P}_\alpha$ the maximal parabolic subgroup. Then $\tilde{\mathcal{G}}_P$ will be denoted by $\tilde{\mathcal{G}}_\alpha$, and the map $\tilde{g}^* \to \tilde{g}_\alpha^*$ is denoted by $\tilde{\pi}_\alpha^*.$

**Corollary 3.3.** Let $\mu \in \Lambda$ be regular and $\nu$ be $\alpha$-regular. Then

$$R_{\mu|\nu}^\alpha \circ \gamma_{X,\mu} \cong \gamma_{X,\nu} \circ L\tilde{\pi}_{\alpha}^{(1)*} \circ R\tilde{\pi}_{\alpha}^{(1)}.$$
Recall from [V] that $s_\alpha : \mathfrak{g}_{\text{reg}}^* \to \mathfrak{g}_{\text{reg}}^*$. Let $\Gamma_\alpha \subseteq \tilde{g}^* \times \tilde{g}^*$ be the closure of the graph of $s_\alpha$ and $\mathcal{O}_\alpha \in \text{Coh}(\tilde{g}^* \times \tilde{g}^*)$ the structure sheaf of $\Gamma_\alpha$.

**Proposition 3.4.** Let $\mu \in \Lambda$ be regular and $\nu$ be $\alpha$-regular. Then

$$FM(\mathcal{O}_\alpha) \circ \gamma_{X,\mu} \cong \gamma_{X,\mu} \circ \Theta_{\mu|\nu}.$$  

In order to prove Proposition 3.4, we need a general lemma about Fourier-Mukai transform. Let $f : X \to Y$ be a proper morphism with graph $\Gamma_f \subseteq X \times Y$ and $\Gamma_f^0 \subseteq Y \times X$.

**Lemma 3.5** (Lemma 1.2.2 in [R]). Notations as above. We have

1. $Rf_* \cong FM(\mathcal{O}_{\Gamma_f})$ and $Lf^* \cong FM(\mathcal{O}_{\Gamma_f^0})$;
2. $Rf_* \circ Lf^* \cong FM(\mathcal{O}_{\Gamma_f} \ast \mathcal{O}_{\Gamma_f^0})$;
3. the adjunction morphism $Rf_* \circ Lf^* \to \text{id}$ is induced by the Fourier-Mukai of the following map

$$\Delta_* \mathcal{O}_X \cong Rp_{13*} \mathcal{O}_{\delta X} \to Rp_{13*} \mathcal{O}_{\Gamma_f \times X \cap X \times \Gamma_f^0} \to \mathcal{O}_{\Gamma_f} \ast \mathcal{O}_{\Gamma_f^0}.$$  

**Proof of Proposition 3.4.** Let $\tilde{\pi}_\alpha : \tilde{g}^* \to \mathfrak{g}_\alpha^*$. We have an isomorphism

$$p_{12}^* \mathcal{O}_{\delta \tilde{g}^*} \otimes L p_{23}^* \mathcal{O}_{\delta \tilde{g}^*} \cong \mathcal{O}_{\Gamma_f \times X \cap X \times \Gamma_f^0}.$$  

There is also an exact triangle

$$\mathcal{O}_\Delta \hookrightarrow \mathcal{O}_{\tilde{g}^* \times \tilde{g}^*} \to \mathcal{O}_{\tilde{g}^*}.$$  

These facts combine to yield Proposition 3.4. □

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