3) Class'n of simple alg'c group

Simple $G \supset T \rightsquigarrow \mathfrak{X}(T) \hookrightarrow \mathfrak{t}^*$

Have wt lattice $\Lambda \subset \mathfrak{t}^*$ & <u>root lattice</u> $\Lambda' = \mathrm{Span}_{\mathbb{Z}} R \subset \Lambda$ fin. index ($\mathrm{Span}_{\mathbb{C}} R = \mathfrak{t}^*$)

Thm S: $\Lambda' \subset \mathfrak{X}(T) \subset \Lambda$; the assignment $G \mapsto \mathfrak{X}(T)$ is 1-1 corresp between conn'd G w. $\mathrm{Lie}(G) = \mathfrak{g}$ & lattices between Λ', Λ

Proof: $\Lambda' \subset \mathfrak{X}(T)$ b/c R = nonzero wts of $\mathrm{Ad}: G \to GL(\mathfrak{g})$; $\mathfrak{X}(T) \subset \Lambda$ b/c $\mathfrak{X}(T) = \mathrm{Span}_{\mathbb{Z}}$(wts of faithf. G-rep'n) $\subset \Lambda$.

Existence: $\underline{\Lambda}$ - fin. coll'n of domin. wts $\rightsquigarrow V_{\underline{\Lambda}} = \bigoplus_{\lambda \in \underline{\Lambda}} V(\lambda) \rightsquigarrow G_{\underline{\Lambda}} \subset \overbrace{\prod_{\lambda \in \underline{\Lambda}} GL(V(\lambda))}^{GL_{\underline{\Lambda}} :=}$ w. Lie alg $\mathfrak{g} \hookrightarrow \mathfrak{gl}_{\underline{\Lambda}}$ ... $\mathfrak{X}(T_{\underline{\Lambda}}) = \mathrm{Span}_{\mathbb{Z}}(\underline{\Lambda})$. Pick $\underline{\Lambda} \subset \Lambda_+$ generating a sublattice Λ_1 between $\Lambda', \Lambda \rightsquigarrow$ gr'p corresp. to Λ_1.

Uniqueness: Let G_1, G_2 w. $\mathfrak{X}(T_1) = \mathfrak{X}(T_2)$. G_1, G_2 have faithf reps $\Rightarrow G_i = G_{\underline{\Lambda}_i}$, $i = 1,2$ for some $\underline{\Lambda}_1, \underline{\Lambda}_2$; $\underline{\Lambda} := \underline{\Lambda}_1 \cup \underline{\Lambda}_2$, $G := G_{\underline{\Lambda}}$. $G_{\underline{\Lambda}} \to G_{\underline{\Lambda}_i}$, in fact, surj'ly (via $GL_{\underline{\Lambda}} \twoheadrightarrow GL_{\underline{\Lambda}_i}$) b/c $\mathfrak{g}_{\underline{\Lambda}} \xrightarrow{\sim} \mathfrak{g}_{\underline{\Lambda}_i}$; $T_{\underline{\Lambda}} \twoheadrightarrow T_{\underline{\Lambda}_i} \rightsquigarrow$ pull-back $\mathfrak{X}(T_{\underline{\Lambda}_i}) \hookrightarrow \mathfrak{X}(T_{\underline{\Lambda}})$. But pull-back intertwines $\mathfrak{X}(T_?) \hookrightarrow \mathfrak{t}^*$ and the images coincide w. $\mathrm{Span}_{\mathbb{Z}}(?)$ ($? = \underline{\Lambda}, \underline{\Lambda}_i$). So $\mathfrak{X}(T_{\underline{\Lambda}_i}) \xrightarrow{\sim} \mathfrak{X}(T_{\underline{\Lambda}}) \Leftrightarrow T_{\underline{\Lambda}} \xrightarrow{\sim} T_{\underline{\Lambda}_i}$. We've mentioned $Z_G(T) = T \Rightarrow Z(G) \subset T$ $\forall$ s/simple gr'p G. So $\ker(G_{\underline{\Lambda}} \to G_{\underline{\Lambda}_i}) \subset \ker(T_{\underline{\Lambda}} \to T_{\underline{\Lambda}_i}) = \{1\}$, so $G_{\underline{\Lambda}} \to G_{\underline{\Lambda}_i}$ is bij've $\Rightarrow$ isom'm $\square$

Rem: $\mathfrak{X}(T)/\Lambda' = Z(G)$, $\Lambda/\mathfrak{X}(T) = \pi_1(G)$