ABELIAN LOCALIZATION FOR CHEREDNIK ALGEBRAS

IVAN LOSEV

1. Introduction

The goal of this talk is to prove an analog of the Beilinson-Bernstein localization theorem for Cherednik algebras. Strictly speaking we will only do this for categories $\mathcal{O}$, in fact, the localization theorem for all modules follows from here.

Let us recall the notation and some definitions. We consider the reflection representation $\mathfrak{h}$ of the symmetric group $\mathfrak{S}_n$. By $X$ we denote the “normalized” Hilbert scheme $X$, a resolution of $X_0 := (\mathfrak{h} \oplus \mathfrak{h}^*)/\mathfrak{S}_n$, we write $\pi$ for the Hilbert-Chow morphism $X \to X_0$. The structure sheaf $\mathcal{O}_X$ has a two-parametric deformation $\mathfrak{A}_{un}$; a sheaf of algebras over $\mathbb{C}[z, \hbar]$, where $z$ is a parameter of a commutative deformation $X$. We have a two-dimensional torus $T_\hbar \times T_c$ ($\hbar$ for “Hamiltonian”, $c$ for “contracting”) acting on $X$ and on $\mathfrak{A}_{un}$.

We can consider the specialization $\mathfrak{A}_\lambda$ of $\mathfrak{A}_{un}$ to $\hbar = 1, z = \lambda$. It still carries a $T_c$-action.

We also consider the algebra $S(\mathfrak{h} \oplus \mathfrak{h}^*)\# \mathfrak{S}_n$ and its two-parametric deformation $H_{un}$ over $\mathbb{C}[c, t]$ that again comes equipped with a $T_\hbar \times T_c$-action. We consider the specialization $H_{1,c}$ of $H_{un}$ to numerical parameters.

Let $\mathcal{P}$ denote the Procesi bundle on $X$, a $T_\hbar \times T_c$-equivariant vector bundle constructed in Gufang’s talk. Let $\mathcal{P}_h$ be its deformation to a right $\mathfrak{A}_{un}$-module (note that this is different from the previous lecture, where we used a deformation to a left module). We write $H^\text{loc}_{un}$ for its endomorphism sheaf. This is a sheaf of $\mathbb{C}[z, \hbar]$-algebras or of $\mathbb{C}[c, t]$-algebras, where $c \mapsto -z, t \mapsto \hbar$ (well, there is another choice of the map, and I’m not 100% sure what one needs to take...). As we have seen in the previous lecture, $\Gamma'(H^\text{loc}_{un}) = H_{un}$, a $T_\hbar \times T_c$-equivariant isomorphism of $\mathbb{C}[c, t]$-algebras. We remark that $\mathcal{P}_h \otimes_{\mathfrak{A}_{un}} \bullet$ is an equivalence of $\text{Coh}(\mathfrak{A}_{un})$ and $\text{Coh}(H^\text{loc}_{un})$. We also have functors $\Gamma : \text{Coh}(H^\text{loc}_{un}) \to H_{un}$-mod of taking global sections and $\text{Loc} : H_{un}$-mod $\to \text{Coh}(H^\text{loc}_{un})$, the localization functor given by $N \mapsto H^\text{loc}_{un} \otimes H_{un} N$. We also consider the specializations of these functors to numerical parameters. We have similar functors between $\text{Coh}(H^{loc}_{1,c}), H_{1,c}$-mod. Let us denote those by $\Gamma_{1,c}, \text{Loc}_{1,c}$, they are obtained from $\Gamma, \text{Loc}$ by specialization to numerical values of parameters. We also remark that $R\Gamma$ and $L\text{Loc}$ are mutually inverse derived equivalences. This was basically established in the previous lecture (with slightly different functors).

**Theorem 1.1.** Suppose $c$ is not of the form $\frac{l}{m}$ with $1 < m \leq n$ and $r < 0$. Then $\Gamma_{1,c}, \text{Loc}_{1,c}$ are mutually inverse equivalences.

We will only prove this theorem for categories $\mathcal{O}$ (this is the most interesting case anyway, and the general one follows from this). Let us recall that by the category $\mathcal{O}$ for $H^{loc}_{1,c}$ we mean the full subcategory of $\text{Coh}(H^{loc}_{1,c})$ consisting of all objects supported on $\pi^{-1}(\mathfrak{h}/\mathfrak{S}_n)$ that have a $T_\hbar$-equivariant structure (under the equivalence $\text{Coh} H^{loc}_{1,c} \to \text{Coh} \mathfrak{A}_\lambda$, this becomes the category $\mathcal{O}$ for $\mathfrak{A}_\lambda$ introduced in the previous lecture). For technical reasons we will only consider $c > 0$ (if $c$ satisfies neither this nor the conditions of the theorem, then the category $\mathcal{O}$ is semisimple (=boring)).
Let us provide some references for the previous theorem. It was first proved by Gordon and Stafford, [GS], in the context of so-called \( \mathbb{Z} \)-algebras (noncommutative analogs of homogeneous coordinate rings). Then the theorem was proved by Kashiwara and Rouquier in the setting close to ours, [KR]. Both approaches had a restriction on parameters: \( c \not\in \frac{1}{2} + \mathbb{Z} \) that was later removed in [BE]. I am going to explain my own proof, [L], that has an advantage to generalize to so-called cyclotomic Rational Cherednik algebras.

2. CORRESPONDENCE BETWEEN VERMA MODULES

We will need two facts mentioned in Kostya’s lecture:

1. \( \mathcal{P}^* \) is flat over \( S(\frak{h}) \), in other words, a basis \( y_1, \ldots, y_n \) of \( \frak{h} \) forms a regular sequence in \( \mathcal{P}^* \).

2. Let \( e_\lambda \) stand for a primitive idempotent corresponding to an irreducible representation \( \lambda \) of \( \mathfrak{S}_n \). Then \([\mathcal{P}^*/\mathcal{P}^*\frak{h}]e_\lambda\) is supported on the union of the contracting components (for the \( T_\frak{h} \)-action) \( Y_\mu \) with \( \mu \leq \lambda \) (well, Kostya had something about \( \mathcal{P} \) and \( \frak{h}^* \) and we are dealing with \( \mathcal{P}^* \) and \( \frak{h} \), but those are similar things).

Recall that Jose has defined Verma modules over \( H_{1,c} \). We can use the same construction for \( H_{un} \): we get \( \Delta_{un}(\lambda) := H_{un} \otimes_{S(\frak{h})} \mathcal{P} \lambda \) \( \lambda = [H_{un}/H_{un}\frak{h}]e_\lambda \) (let us note that \( H_{un}/H_{un}\frak{h} = H_{un} \otimes_{S(\frak{h})} \mathcal{P} \mathfrak{S}_n \)). This module is flat over \( \mathbb{C}[c, t] \), its specialization to \( 1, c \) gives \( \Delta_{1,c}(\lambda) \).

Similarly, we can define the Verma modules for \( H_{loc} \). \( \Delta_{loc}(\lambda) = [H_{loc}/H_{loc}\frak{h}]e_\lambda \). Similarly, we can define their specializations \( \Delta_{loc}^{1,c}(\lambda) \).

The following proposition establishes some properties of \( \Delta_{loc}^{1,c}(\lambda) \) based on (i).

**Proposition 2.1.** The following is true:

1. \( H_{loc}^{1,c} \) is flat as a right module over \( S(\frak{h})[c, t] \).
2. \( \Delta_{loc}^{1,c}(\lambda) \) is flat as a \( \mathbb{C}[c, t] \)-module.
3. \( \Delta_{loc}^{1,c}(\lambda) = \mathcal{P} \otimes [\mathcal{P}^*/\mathcal{P}^*\frak{h}]e_\lambda \) and so has the support as specified in (ii).
4. We have \( RT(\Delta_{loc}^{1,c}(\lambda)) = \Delta_{un}(\lambda) \) (meaning that the usual global sections are as specified and the higher derived sections vanish).
5. We have \( L\text{Loc}(\Delta_{un}(\lambda)) = \Delta_{loc}^{1,c}(\lambda) \).

**Proof.** Because of the \( T_\frak{h} \)-equivariance and the claim that \( H_{loc}^{1,c} \) is a flat over \( \mathbb{C}[c, t] \) deformation of \( \text{End}(\mathcal{P}) \), it is enough to show that \( \text{End}(\mathcal{P}) \) is flat as a right module over \( S(\frak{h}) \), but this follows from (i). So (1) is proved. (2) is a corollary of (1) (an exercise). (3) is a direct consequence of the definition of \( \Delta_{loc}^{1,c}(\lambda) \).

Let us prove (4), (5) is given as an exercise. Because of (i), as an object of \( D^b(\text{Coh}^Tc H_{loc}^{1,c}) \), \( \Delta_{loc}^{1,c}(\lambda) \) is \( K(H_{loc}^{1,c}, \frak{h})e_\lambda \), where \( K(H_{loc}^{1,c}, \frak{h}) \) is the Koszul complex:

\[
H_{loc}^{1,c} \otimes K^2 \frak{h} \to H_{loc}^{1,c} \otimes \frak{h} \to H_{loc}^{1,c}
\]

All terms in \( K(H_{loc}^{1,c}, \frak{h}) \) are acyclic for \( RT \). Indeed, \( RT(\Delta_{loc}^{1,c}(\lambda)) = H_{loc}^{1,c}(\Delta_{loc}^{1,c}(\lambda)) = 0 \) for \( i > 0 \) (because \( H_{loc}^{1,c} \) is a flat \( T_\frak{h} \)-equivariant deformation of \( \text{End}(\mathcal{P}) \), where the cohomology vanishing holds). So \( RT(\Delta_{loc}^{1,c}(\lambda)) \) is given by the complex \( \Gamma(K(H_{loc}^{1,c}, \frak{h})e_\lambda) \). But \( \Gamma(H_{loc}^{1,c} \otimes K^2 \frak{h}) = \Gamma(H_{loc}^{1,c} \otimes K^1 \frak{h} = H_{loc}^{1,c} \otimes K^1 \frak{h} \). So \( \Gamma(K(H_{loc}^{1,c}, \frak{h})) = K(H_{loc}^{1,c}, \frak{h}) \). We deduce that \( RT(\Delta_{loc}^{1,c}(\lambda)) = K(H_{loc}^{1,c}, \frak{h})e_\lambda \) which is a resolution of \( \Delta_{un}(\lambda) \). (4) is proved.

3. LOCALIZATION THEOREM

Now we are ready to prove Theorem 1.1 for categories \( \mathcal{O} \). Recall that \( \mathcal{C} := \mathcal{O}(H_{1,c}) \) is a highest weight category. There is an additional structure: the labeling set of its simples
– the set of partitions of \( n \) to be denoted by \( P(n) \) is equipped with a partial order given by \( \lambda < \mu \) if \( c(\mathrm{cont}(\mu) - \mathrm{cont}(\lambda)) \in \mathbb{Z}_{\geq 0} \). Then there are axioms. First, \( \mathcal{C} \) is a length category, with finitely many simples, finite dimensional Hom’s and enough projectives. Second, there are standard objects (that happen to coincide with Verma modules) \( \Delta(\lambda) \) satisfying upper triangularity conditions explained in Jose’s lecture. One consequence of those conditions is that standard objects are uniquely determined from the order: namely, we call \( \mathcal{C} \) a standard dominance order on the set of partitions. The category \( \mathcal{C} \) consists of all objects supported on \( \bigsqcup_{\mu \leq \lambda} Y_{\mu} \).

Now the proof of Theorem 1.1 is based on two statements.

**Proposition 3.1.** Suppose that \( c > 0 \). Then the object \( \Delta_{1,c}(\lambda) \) is a standard object in \( \mathcal{C}' \) corresponding to \( \lambda \).

**Proof.** The support of \( \Delta_{1,c}^{\text{loc}}(\lambda) \) is the same thing as that of \( \Delta_{1,c}^{\text{loc}}(\lambda) \). So \( \Delta_{1,c}^{\text{loc}}(\lambda) \) lies in \( \mathcal{C}_{<\lambda} \) by (ii) and (3) of the lemma above. Now we can argue by the ascending induction on \( \lambda \). Suppose we know our claim for all \( \mu < \lambda \), i.e., \( \Delta_{1,c}^{\text{loc}}(\mu) = \Delta'(\mu) \). Recall that \( R\Gamma_{1,c}(\Delta_{1,c}^{\text{loc}}(\lambda)) = \Delta_{1,c}(\lambda) \) and that \( R\Gamma \) is a derived equivalence. We have \( \mathrm{RHom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) = 0 \) (exercise: why?) and hence \( \mathrm{RHom}_{\mathcal{C}'}(\Delta_{1,c}^{\text{loc}}(\lambda), \Delta_{1,c}^{\text{loc}}(\mu)) = 0 \). For similar reason, \( \mathrm{RHom}_{\mathcal{C}'}(\Delta_{1,c}^{\text{loc}}(\lambda), \Delta_{1,c}^{\text{loc}}(\lambda)) = \mathbb{C} \) (in homological degree 0). It follows from the inductive assumption that \( \Delta_{1,c}^{\text{loc}}(\lambda) \) is an indecomposable projective not lying in \( \mathcal{C}_{<\lambda} \), and so we are done.

The second part of the proof is a pure abstract nonsense. Let us observe that if we refine a highest weight order, it is still a highest weight order (upper triangularity properties become just less restrictive). Under the conditions of the theorem (this is where we use the assumption on \( c \)) the orders for our categories are refined by a single order: \( \lambda < \mu \) if \( \mathrm{cont}(\lambda) < \mathrm{cont}(\mu) \). And now the abstract nonsense.

**Proposition 3.2.** Let \( \mathcal{C}, \mathcal{C}' \) be two highest weight categories whose simples are indexed by the same poset. Let \( \mathcal{F} : \mathcal{C} \simeq \mathcal{C}' : \mathcal{G} \) be a pair of adjoint functors (\( \mathcal{G} \) is right adjoint to \( \mathcal{F} \)). Suppose that \( R\mathcal{G}(\Delta'(\lambda)) = \Delta(\lambda), L\mathcal{F}(\Delta(\lambda)) = \Delta'(\lambda) \). Then \( \mathcal{F}, \mathcal{G} \) are mutually inverse equivalences.

I don’t want to provide the proof. People who’s done exercises on highest weight categories (=Pasha) may want to prove this is a problem (well, not quite an exercise).

This completes the proof of Theorem 1.1.

**References**


