# INTRODUCTION TO RATIONAL CHEREDNIK ALGEBRAS

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Abstract. These are notes for a talk in the MIT-Northeastern Spring 2015 Graduate Representation Theory Seminar. The main source is [BR14].

Contents



## 1. Definition of rational Cherednik algebra

1.1. **Complex reflection groups.** A group $W$ is a complex reflection group if it is equipped with a *reflection representation* $V$ of dimension $n$ and generated by a set of reflections $\{s_1, \ldots, s_d\}$ for which

$$\dim(s_i - \mathrm{id}_V) = 1.$$

Let $\mathrm{Ref}(W)$ denote the set of reflections of $W$, and let $\varepsilon : W \to \mathbb{C}^\times$ be the composition $W \to GL(V) \overset{\det}{\to} \mathbb{C}^\times$. For $s \in \mathrm{Ref}(W)$, choose elements $\alpha_s \in V^*$ and $\alpha_s^\vee \in V$ so that

$$\mathrm{Im}(s - \mathrm{id}_V) = \mathbb{C} \cdot \alpha_s^\vee \qquad \mathrm{Im}(s - \mathrm{id}_{V^*}) = \mathbb{C} \cdot \alpha_s.$$

Note that this implies $\ker(s - \mathrm{id}_V) = \ker(\alpha_s)$ and $\ker(s - \mathrm{id}_{V^*}) = \ker(\alpha_s^\vee)$.

1.2. **Invariants of complex reflection groups.** The ring of functions $\mathbb{C}[V]$ admits a representation of $W$. Its $W$-invariants admit the following description.

**Theorem 1.1** (Shephard-Todd, Chevalley)**.** The algebra of invariants $\mathbb{C}[V]^W$ is a polynomial ring generated by homogeneous elements of degree $d_1, \ldots, d_n$ so that

$$|W| = d_1 \cdots d_n \text{ and } |\mathrm{Ref}(W)| = \sum_i (d_i - 1).$$

*Date*: February 4, 2015.

1.3. **The definition of the rational Cherednik algebra.** Let $\mathcal{C}$ be the vector space of maps $\mathrm{Ref}(W) \to \mathbb{C}$ which are constant on conjugacy classes, and let $\widetilde{\mathcal{C}} = \mathbb{C} \times \mathcal{C}$. This implies that

$$\mathbb{C}[\widetilde{\mathcal{C}}] = \mathbb{C}[T, (C_s)_{s \in \mathrm{Ref}(W)/W}],$$

where $T$ is projection to $\mathbb{C}$ and $C_s$ evaluation at $s \in \mathrm{Ref}(W)$.

**Definition 1.2.** The generic rational Cherednik algebra is the $\mathbb{C}[\widetilde{\mathcal{C}}]$-algebra $\mathbf{H}$ which is the quotient of

$$\mathbb{C}[\widetilde{\mathcal{C}}] \otimes T(V \oplus V^*) \rtimes \mathbb{C}[W]$$

by the relations

$$[x, x'] = [y, y'] = 0$$

$$[y, x] = T\langle y, x\rangle + \sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s) - 1) C_s \frac{\langle y, \alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s,$$

for $x, x' \in V^*$ and $y, y' \in V$.

1.4. **Specialization of Cherednik algebras.** For $(t, c) \in \widetilde{\mathcal{C}}$, define the specialization $\mathbf{H}_{t,c}$ of $\mathbf{H}$ to be

$$\mathbf{H}_{t,c} := \mathbb{C}_{t,c} \underset{\mathbb{C}[\widetilde{\mathcal{C}}]}{\otimes} \mathbf{H}$$

where $\mathbb{C}[\widetilde{\mathcal{C}}] \to \mathbb{C}_{t,c}$ is given by evaluation at $(t, c)$. If $c = 0$, we recover the trivial examples

$$\mathbf{H}_{0,0} = \mathbb{C}[V \oplus V^*] \rtimes \mathbb{C}[W] \qquad \text{and} \qquad \mathbf{H}_{t,0} = \mathcal{D}_t(V) \rtimes \mathbb{C}[W],$$

where $\mathcal{D}_t(V)$ denotes the ring of differential operators on $V$, defined as the quotient of $\mathbb{C}[V \oplus V^*]$ by

$$[x, x'] = 0 \qquad [y, y'] = 0 \qquad [y, x] = t\langle y, x\rangle.$$

Define also $\mathcal{D}_T(V)$ to be the $\mathbb{C}[T]$ algebra given as the quotient of $\mathbb{C}[T] \otimes \mathbb{C}[V \oplus V^*]$ by

$$[x, x'] = 0 \qquad [y, y'] = 0 \qquad [y, x] = T\langle y, x\rangle.$$

## 2. Basic properties of Cherednik algebras

2.1. **Filtration on H.** We define a filtration on $\mathbf{H}$ by

- $\mathbf{H}^{\leq -1} = 0$;
- $\mathbf{H}^{\leq 0} = \mathbb{C}[\widetilde{\mathcal{C}}] \cdot \mathbb{C}[V^*] \cdot \mathbb{C}[W]$;
- $\mathbf{H}^{\leq 1} = \mathbf{H}^{\leq 0} \cdot V + \mathbf{H}^{\leq 0}$;
- $\mathbf{H}^{\leq i} = (\mathbf{H}^{\leq 1})^i$ for $i \geq 2$.

2.2. **Dunkl operators and the PBW theorem.** Let $V^{\mathrm{reg}} = V - \bigcup_H H = \{v \in V \mid \mathrm{Stab}_W(v) = 1\}$ so that $\mathbb{C}[V^{\mathrm{reg}}] = \mathbb{C}[V][\delta^{-1}]$. This implies that $\mathcal{D}_T(V^{\mathrm{reg}}) = \mathcal{D}_T(V)[\delta^{-1}]$. Denote by $\mathbf{H}^{\mathrm{reg}} := \mathbf{H}[\delta^{-1}]$. For $y \in V$, the *Dunkl operator* $D_y$ is the $\mathbb{C}[\widetilde{\mathcal{C}}]$-linear endomorphism of $\mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V]$ given by

$$D_y = T\partial_y - \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \langle y, \alpha_s\rangle \alpha_s^{-1}(s - 1) \in \mathbb{C}[\mathcal{C}] \otimes \mathcal{D}_T(V^{\mathrm{reg}}) \rtimes \mathbb{C}[W].$$

These operators yield a representation of $\mathbf{H}$ on $\mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V]$.

**Proposition 2.1.** There is a representation of $\mathbf{H}$ on $\mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V]$ where $V^*$ acts by multiplication, $V$ acts by Dunkl operators, and $W$ acts by the representation action on $V$.

*Proof.* It suffices to check commutation relations involving elements of $V$. For $y \in V$ and $x \in V^*$, notice that

$$[\alpha_s^{-1} s, x] = (\varepsilon(s)^{-1} - 1) \frac{\langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s$$

and therefore

$$[D_y, x] = T\langle y, x\rangle + \sum_s (\varepsilon(s) - 1) C_s \frac{\langle y, \alpha_s\rangle \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s.$$

By checking directly, we see that $wD_y w^{-1} = D_{w(y)}$. Finally, for $y, y' \in V$, we have that

$$[[D_y, D_{y'}], x] = [[D_y, x], D_{y'}] - [[D_{y'}, x], D_y],$$

where we have

$$\begin{aligned}[[D_y, x], D_{y'}] &= \sum_s (\varepsilon(s) - 1) C_s \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} [s, D_{y'}] \\ &= \sum_s (\varepsilon(s) - 1)^2 C_s \frac{\langle y, \alpha_s \rangle \langle y', \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle^2} D_{\alpha_s^\vee} s \\ &= [[D_{y'}, x], D_y],\end{aligned}$$

which implies that $[D_y, D_{y'}]$ commutes with $\mathbb{C}[V]$. On the other hand, $[D_y, D_{y'}]$ acts by 0 on $1 \in \mathbb{C}[V]$, hence by 0 on all of $\mathbb{C}[V]$. Because the action of $\mathcal{D}_T(V^{\text{reg}})$ on $\mathbb{C}[V]$ is faithful, this implies $[D_y, D_{y'}] = 0$, completing the proof. □

**Remark.** This action is called the polynomial representation of $\mathbf{H}$.

By analyzing the polynomial representation, we are able to obtain a PBW theorem for $\mathbf{H}$.

**Proposition 2.2.** The linear map

$$\mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V] \otimes \mathbb{C}[W] \otimes \mathbb{C}[V^*] \overset{\text{mult}}{\to} \mathbf{H}$$

is an isomorphism of $\mathbb{C}[\widetilde{\mathcal{C}}]$-modules.

*Proof.* The polynomial representation yields a map

$$\Theta : \mathbf{H} \to \mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathcal{D}_T(V^{\text{reg}}).$$

Denote by $\Theta^{\text{reg}}$ the extension to $\mathbf{H}^{\text{reg}} \to \mathbb{C}[\mathcal{C}] \otimes \mathcal{D}_T(V^{\text{reg}})$. Consider the composition

$$\eta : \mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V^{\text{reg}}] \otimes \mathbb{C}[W] \otimes \mathbb{C}[V^*] \overset{\text{mult}^{\text{reg}}}{\to} \mathbf{H}^{\text{reg}} \overset{\Theta^{\text{reg}}}{\to} \mathbb{C}[\mathcal{C}] \otimes \mathcal{D}_T(V^{\text{reg}}) \rtimes \mathbb{C}[W].$$

Notice that $\text{gr}(\eta)$ is an isomorphism, hence $\eta$ is an isomorphism. Now, because $\text{mult}^{\text{reg}}$ is surjective by definition, this implies that it and $\Theta^{\text{reg}}$ are both injections. This implies by restriction that the map

$$\mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V] \otimes \mathbb{C}[W] \otimes \mathbb{C}[V^*] \overset{\text{gr(mult)}}{\to} \text{gr}\mathbf{H}$$

is injective, hence an isomorphism, which yields the desired. □

**Corollary 2.3.** The polynomial representation is faithful.

*Proof.* The proof of Proposition 2.2 also shows that $\Theta$ is injective by restriction from the isomorphism $\Theta^{\text{reg}}$. Faithfulness follows because the map $\mathcal{D}_T(V^{\text{reg}}) \rtimes \mathbb{C}[W] \to \mathbb{C}[\widetilde{\mathcal{C}}] \otimes \text{Hom}_k(\mathbb{C}[V], \mathbb{C}[V^{\text{reg}}])$ is injective and the image of the polynomial representation under this identification lands in $\mathbb{C}[\widetilde{\mathcal{C}}] \otimes \text{End}_k(\mathbb{C}[V])$. □

2.3. **The center of $\mathbf{H}_{t,c}(W)$ at $t \neq 0$.** Let $\mathcal{Z}$ denote the center of $\mathbf{H}$ and $\mathcal{Z}_{t,c}$ its specialization.

**Proposition 2.4.** If $t \neq 0$, then the polynomial representation of $\mathbf{H}_{t,c}$ is faithful and $\mathcal{Z}_{t,c} = \mathbb{C}$.

*Proof.* Faithfulness follows in the same way as in Corollary 2.3, where we note that the polynomial representation of $\mathcal{D}_t(V) \rtimes \mathbb{C}[W]$ is faithful only when $t \neq 0$.

Now, by faithfulness, the polynomial representation gives an embedding $\mathbf{H}_{t,c}(W) \hookrightarrow \mathcal{D}_t(V) \rtimes \mathbb{C}[W] \simeq \mathcal{D}(V) \rtimes \mathbb{C}[W]$ for $t \neq 0$. Any element of $\mathcal{Z}_{t,c}$ must commute with $\mathbb{C}[V^*] \subset \mathcal{D}(V) \rtimes \mathbb{C}[W]$, hence lie in $\mathbb{C}[V^*]$. It is easy to check that no non-constant element of $\mathbb{C}[V^*]$ commutes with all $x \in V^*$, showing that $\mathcal{Z}_{t,c} = \mathbb{C}$. □

2.4. **The spherical Cherednik algebra.** The primitive central idempotent of $\mathbb{C}[W]$ is

$$e = \frac{1}{|W|} \sum_{w \in W} w,$$

and the $\mathbb{C}[\widetilde{\mathcal{C}}]$-algebra $e\mathbf{H}e$ is known as the *generic spherical algebra*. We denote its specialization by $e\mathbf{H}_{t,c}e$. By Proposition 2.2, we see that

$$\text{gr}(e\mathbf{H}e) = \mathbb{C}[\widetilde{\mathcal{C}}] \otimes \mathbb{C}[V \oplus V^*]^W.$$

We first examine a few properties of the spherical algebra.

**Proposition 2.5.** The following properties hold:

(a) $e\mathbf{H}_{t,c}e$ is a finitely generated $\mathbb{C}$-algebra without zero divisors;
(b) $\mathbf{H}_{t,c}e$ is a finitely generated right $e\mathbf{H}_{t,c}e$-module;
(c) left multiplication yields an isomorphism $\mathbf{H}_{t,c} \to \mathrm{End}_{(e\mathbf{H}_{t,c}e)^{\mathrm{op}}}(\mathbf{H}_{t,c}e)^{\mathrm{op}}$;

*Proof.* Properties (a) and (b) follow because they hold for $\mathrm{gr}(e\mathbf{H}_{t,c}e)$. For property (b), let $\phi : \mathbf{H}_{t,c} \to \mathrm{End}_{(e\mathbf{H}_{t,c}e)^{\mathrm{op}}}(\mathbf{H}_{t,c}e)^{\mathrm{op}}$ be the desired morphism. We consider the composition

$$\psi : \mathrm{gr}\mathbf{H}_{t,c} \overset{\mathrm{gr}(\phi)}{\to} \mathrm{grEnd}_{(e\mathbf{H}_{t,c}e)^{\mathrm{op}}}(\mathbf{H}_{t,c}e)^{\mathrm{op}} \to \mathrm{End}_{\mathrm{gr}(e\mathbf{H}_{t,c}e)^{\mathrm{op}}}(\mathrm{gr}(\mathbf{H}_{t,c}e))^{\mathrm{op}},$$

where the first map is by left multiplication and the second is an injection[1]. Recall from the proof of Proposition 2.2 the isomorphism $\mathrm{gr}\mathbf{H}_{t,c} \simeq \mathbb{C}[V \oplus V^*] \rtimes \mathbb{C}[W]$, under which this map is given by left multiplication

$$\mathbb{C}[V \oplus V^*] \rtimes \mathbb{C}[W] \to \mathrm{End}_{(e(\mathbb{C}[V\oplus V^*]\rtimes\mathbb{C}[W])e)^{\mathrm{op}}}((\mathbb{C}[V \oplus V^*] \rtimes \mathbb{C}[W])e)^{\mathrm{op}},$$

hence is an isomorphism by Lemma 2.6 applied to $X = V \times V^*$ with the action of $W$, where the codimension condition holds because the action of $W$ preserves the pairing between $V$ and $V^*$. We conclude that $\psi$ is an isomorphism, hence $\mathrm{gr}(\phi)$ and $\phi$ are, as needed. □

**Lemma 2.6.** Let $W$ act on a smooth affine variety $X$, let $A = \mathbb{C}[X]$, and let $R = A \rtimes \mathbb{C}[W]$. Let $X^{\mathrm{reg}} = \{x \in X \mid \mathrm{Stab}_W(x) = 1\}$. If $\mathrm{codim}(X - X^{\mathrm{reg}}) \geq 2$ in each connected component, then the morphism $R \to \mathrm{End}_{(A^W)^{\mathrm{op}}}(A)^{\mathrm{op}}$ is an isomorphism.

*Proof.* We claim that the morphism is injective even if the codimension condition does not hold. Because $X^{\mathrm{reg}}$ is Zariski dense, we may localize to $\mathbb{C}[X^{\mathrm{reg}}]$ to check injectivity, so we may assume that $W$ acts freely on $X$. In this case, choose $\sum_i f_i \otimes w_i$ in the kernel so that $\sum_i f_i w_i(f) = 0$ for $f \in A$. Because $W$ acts freely on $X$, for any $x$ and $z_i \in \mathbb{C}$ we may find some function $f \in A$ so that $f(w_i^{-1} \cdot x) = z_i$, meaning that $\sum_i f_i(x) z_i = 0$, whence we conclude $f_i = 0$, yielding injectivity.

If $W$ acts freely on all of $X$, $R$ and $\mathrm{End}_{(A^W)^{\mathrm{op}}}(A)^{\mathrm{op}}$ are both $A^W$-algebras of rank $|W|^2$, so injectivity implies surjectivity. For surjectivity in general, for any $f \in \mathrm{End}_{(A^W)^{\mathrm{op}}}(A)^{\mathrm{op}}$, cover $X^{\mathrm{reg}}$ by affine open sets $X^j$. On each $X^j$, we may choose some $\sum_i a_i^j \cdot w_i \in \mathbb{C}[X^{\mathrm{reg}}] \rtimes \mathbb{C}[W]$ with $a_i^j \in \mathbb{C}[X^j]$ and $w_i \in W$ which gives rise to the restriction of $f$ to $X^j$. On $X^{j_1} \cap X^{j_2}$, the restriction of $\sum_i a_i^{j_1} \cdot w_i$ and $\sum_i a_i^{j_2} \cdot w_i$ gives rise to the restriction of $f$ to $X^{j_1} \cap X^{j_2}$, hence their restrictions are equal. Therefore, the family of functions $\{a_i^j\}$ glue to a function $a_i$ on $X^{\mathrm{reg}}$ for which $\sum_i a_i \cdot w_i \in \mathbb{C}[X^{\mathrm{reg}}] \rtimes \mathbb{C}[W]$ gives rise to $f|_{X^{\mathrm{reg}}}$. Each $a_i$ is regular in codimension 2, hence regular by Hartog's theorem. Thus $\sum_i a_i \cdot w_i$ lies in $R$, finishing the proof. □

2.5. **The Satake isomorphism.** For the rest of the talk, we work in the specialization $t = 0$. Our goal will be to prove the Satake isomorphism relating $\mathcal{Z}_{0,c}$ and $e\mathbf{H}_{0,c}e$.

**Theorem 2.7** (Satake isomorphism)**.** The map $z \mapsto z \cdot e$ is an isomorphism of algebras $\mathcal{Z}_{0,c} \to e\mathbf{H}_{0,c}e$.

**Lemma 2.8.** If $e$ is an idempotent of a ring $A$ and left multiplication gives an isomorphism

$$A \to \mathrm{End}_{(eAe)^{\mathrm{op}}}(Ae)^{\mathrm{op}},$$

then the map $Z(A) \to Z(eAe)$ given by $a \mapsto ae$ is an isomorphism.

*Proof.* Notice that we have $eAe = \mathrm{End}_A(Ae)$ by definition. Therefore, left multiplication on $Ae$ yields a map $\alpha : Z(A) \to Z(eAe)$ so that $\alpha(z) = ze$ implies $zm = m\alpha(z)$ and by the given right multiplication yields a map $\beta : Z(eAe) \to Z(A)$ so that $mz = \beta(z)m$. For $z \in Z(A)$, we then have that $zm = \beta(\alpha(z))m$, so that $\beta \circ \alpha = \mathrm{id}$ because the left multiplication is faithful. Similarly, we find that $\alpha \circ \beta = \mathrm{id}$. □

*Proof of Theorem 2.7.* By Proposition 2.5(c) and Lemma 2.8, we have $\mathcal{Z}_{0,c} \simeq \mathcal{Z}(e\mathbf{H}_{0,c}e)$, so we only need show $e\mathbf{H}_{0,c}e$ is commutative. The Dunkl operators at $t = 0$ yield an injection $\mathbf{H}_{0,c} \to \mathbb{C}[V^{\mathrm{reg}} \oplus V^*] \rtimes \mathbb{C}[W]$ which restricts to an injection $e\mathbf{H}_{0,c}e \to \mathbb{C}[V^{\mathrm{reg}} \oplus V^*]^W$ with the latter commutative, yielding the claim. □

## References

[BR14] Cédric Bonnafé and Raphaël Rouquier. Calogero-Moser cells. 2014.

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[1] If $M$ and $N$ are filtered modules, over a filtered ring $A$, we filter $\mathrm{Hom}_A(M,N)$ by $\mathrm{Hom}_A(M,N)^{\leq i} = \{f \in \mathrm{Hom}_A(M,N) \mid f(M^{\leq j}) \subset N^{\leq j+i}\}$. There is a map $\mathrm{grHom}_A(M,N) \to \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(M), \mathrm{gr}(N))$ which sends $[f_i] \in \mathrm{gr}^i\mathrm{Hom}_A(M,N)$ to $\Big([m^j] \mapsto [f_i(m^j)] \in \mathrm{gr}^{i+j}(N)\Big)$. We apply this construction with $A = e\mathbf{H}_{t,c}e$ and $M = N = \mathbf{H}_{t,c}e$.