SCHUR-WEYL DUALITY FOR QUANTUM GROUPS

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Abstract. These are notes for a talk in the MIT-Northeastern Fall 2014 Graduate seminar on Hecke algebras and affine Hecke algebras. We formulate and sketch the proofs of Schur-Weyl duality for the pairs \((U_q(\mathfrak{sl}_n), H_q(m))\), \((Y(\mathfrak{sl}_n), \Lambda_m)\), and \((U_q(\hat{\mathfrak{sl}}_n), H_q(m))\). We follow mainly \([\text{Ara99, Jim86, Dri86, CP96}]\), drawing also on the presentation of \([\text{BGHP93, Mol07}]\).

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1. Introduction

Let \(V = \mathbb{C}^n\) be the fundamental representation of \(\mathfrak{sl}_n\). The vector space \(V^\otimes m\) may be viewed as a \(U(\mathfrak{sl}_n)\) and \(S_m\)-representation, and the two representations commute. Classical Schur-Weyl duality gives a finer understanding of this representation. We first state the classifications of representations of \(S_m\) and \(\mathfrak{sl}_n\).

Theorem 1.1. The finite dimensional irreducible representations of \(S_m\) are parametrized by partitions \(\lambda \vdash m\). For each such \(\lambda\), the corresponding representation \(S_\lambda\) is called a Specht module.

Theorem 1.2. The finite dimensional irreducible representations of \(\mathfrak{sl}_n\) are parametrized by signatures \(\lambda\) with \(\ell(\lambda) \leq n\) and \(\sum_i \lambda_i = 0\). For any partition \(\lambda\) with \(\ell(\lambda) \leq n\), there is a unique shift \(\lambda'\) of \(\lambda\) so that \(\sum_i \lambda'_i = 0\). We denote the irreducible with this highest weight by \(L_\lambda\).

The key fact underlying classical Schur-Weyl duality is the following decomposition of a tensor power of the fundamental representation.

Theorem 1.3. View \(V^\otimes m\) as a representation of \(S_m\) and \(U(\mathfrak{sl}_n)\). We have the following:

(a) the images of \(\mathbb{C}[S_m]\) and \(U(\mathfrak{sl}_n)\) in \(\text{End}(W)\) are commutants of each other, and
Lemma 1.4. The irreducible $L_\lambda$ is of weight $m \leq n - 1$ if and only if $\lambda = \sum_i c_i \omega_i$ with $\sum_i i c_i = m$.

We now reframe this result as a relation between categories of representations; this reformulation will be the one which generalizes to the affinized setting. Say that a representation of $U(\mathfrak{s}_n)$ is of weight $m$ if each of its irreducible components occurs in $V^\otimes m$. In general, the weight of a representation is not well-defined; however, for small weight, we have the following characterization from the Pieri rule.

Theorem 1.5. For $n > m$, the functor $\text{FS}$ is an equivalence of categories between $\text{Rep}(S_m)$ and the subcategory of $\text{Rep}(U(\mathfrak{s}_n))$ consisting of weight $m$ representations.

In this talk, we discuss generalizations of this duality to the quantum group setting. In each case, $U(\mathfrak{s}_n)$ will be replaced with a quantization ($U_q(\mathfrak{sl}_n)$, $Y_h(\mathfrak{s}_n)$, or $U_q(\mathfrak{sl}_n)$), and $\mathbb{C}[S_m]$ will be replaced by a Hecke algebra ($H_q(m)$, $\Lambda_m$, or $\mathcal{H}_q(m)$).

2. Finite-type quantum groups and Hecke algebras

2.1. Definition of the objects. Our first generalization of Schur-Weyl duality will be to the finite type quantum setting. In this case, $U_q(\mathfrak{sl}_n)$ will replace $U(\mathfrak{s}_n)$, and the Hecke algebra $H_q(m)$ of type $A_{m-1}$ will replace $S_m$. We begin by defining these objects.

Definition 2.1. Let $\mathfrak{g}$ be a simple Kac-Moody Lie algebra of simply laced type with Cartan matrix $A = (a_{ij})$. The Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ is the Hopf algebra given as follows. As an algebra, it is generated by $x_i^\pm$ and $q^{a_i}$ for $i = 1, \ldots, n - 1$ so that $\{q^{a_i}\}$ are invertible and commute, and we have the relations

$$q^{a_i}x_j^\pm q^{-a_i} = q^{a_i}x_j^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} q^{a_i} - q^{-a_i}, \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1 - a_{ij}}{r} \right] (x_i^\pm)^r x_j^- (x_i^\pm)^{1-a_{ij} - r} = 0.$$ 

The coalgebra structure is given by the coproduct

$$\Delta(x_i^+) = x_i^+ \otimes q^{a_i} + 1 \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes 1 + q^{-a_i} \otimes x_i^-,$$

and counit $\varepsilon(x_i^+) = 0$ and $\varepsilon(q^{a_i}) = 1$, and the antipode is given by

$$S(x_i^+) = -x_i^+ q^{-a_i}, \quad S(x_i^-) = -q^{a_i} x_i^-,$$

$$S(q^{a_i}) = q^{-a_i}.$$

Definition 2.2. The Hecke algebra $H_q(m)$ of type $A_{m-1}$ is the associative algebra given by

$$H_q(m) = \langle T_1, \ldots, T_{m-1} | (T_i - q^{-1})T_i = 0, T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, [T_i, T_j] = 0 \text{ for } |i - j| \neq 1 \rangle.$$

2.2. R-matrices and the Yang-Baxter equation. To obtain $H_q(m)$-representations from $U_q(\mathfrak{sl}_n)$-representations, we use the construction of $R$-matrices.

Proposition 2.3. There exists a unique universal $R$-matrix $\mathcal{R} \in U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{sl}_n)$ such that:

(a) $R \in q \sum x_i \otimes x_j (1 + (U_q(n_+ \otimes U_q(n_-))_{>0})$ for $\{x_i\}$ an orthonormal basis of $\mathcal{R}$, and
(b) $R \Delta(x) = \Delta(2^x) R \mathcal{R}$, and
(c) $(\Delta \otimes 1) R = R^{13} R^{23} \text{ and } (1 \otimes \Delta) R = R^{13} R^{12}.$

We say that such an $\mathcal{R}$ defines a pseudotriangular structure on $U_q(\mathfrak{sl}_n)$. Let $P(x \otimes y) = y \otimes x$ denote the flip map, and let $\mathcal{R} = P \circ \mathcal{R}$. From Proposition 2.3, we may derive several additional properties of $\mathcal{R}$ and $\mathcal{R}$.

Corollary 2.4. The universal $R$-matrix of $U_q(\mathfrak{sl}_n)$:
We explain a proof for

\[ R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}; \]

\( R \) from the definition of \( R \). The braid relation follows from Corollary 2.4(c) and the commutativity of non-adjacent reflections \( R | \) matrix count from the non-quantum case. By the definition of \( R \) we now claim that

\[ \text{Lemma 2.5.} \quad \text{The map } \sigma \text{ satisfies a different version of the Yang-Baxter equation} \]

\[ \hat{R}^{23}\hat{R}^{12}\hat{R}^{23} = \hat{R}^{12}\hat{R}^{23}\hat{R}^{12}; \]

\( \hat{R} \). Obtaining Schur-Weyl duality.

We wish to use Corollary 2.4 to define \( H_q(m) \)-action on \( V^{\otimes m} \). Define the map \( \sigma^m : H_q(m) \to \text{End}(V^{\otimes m}) \) by

\[ \sigma^m : T_i \mapsto \hat{R}^{i,i+1}. \]

\[ \text{Lemma 2.5.} \quad \text{The map } \sigma^m \text{ defines a representation of } H_q(m) \text{ on } V^{\otimes m}. \]

Proof. The braid relation follows from Corollary 2.4(c) and the commutativity of non-adjacent reflections from the definition of \( \sigma^m \). The Hecke relation follows from a direct check on the eigenvalues of the triangular matrix \( R|_{V \otimes V} \) from (1).

\[ \text{Lemma 2.5.} \quad \text{The map } \sigma^m \text{ defines a representation of } H_q(m) \text{ on } V^{\otimes m}. \]

Proof. The braid relation follows from Corollary 2.4(c) and the commutativity of non-adjacent reflections from the definition of \( \sigma^m \). The Hecke relation follows from a direct check on the eigenvalues of the triangular matrix \( R|_{V \otimes V} \) from (1).

2.3. From the Yang-Baxter equation to the Hecke relation. We wish to use Corollary 2.4 to define a \( H_q(m) \)-action on \( V^{\otimes m} \). Define the map \( \sigma^m : H_q(m) \to \text{End}(V^{\otimes m}) \) by

\[ \sigma^m : T_i \mapsto \hat{R}^{i,i+1}. \]

\[ \text{Lemma 2.5.} \quad \text{The map } \sigma^m \text{ defines a representation of } H_q(m) \text{ on } V^{\otimes m}. \]

Proof. The braid relation follows from Corollary 2.4(c) and the commutativity of non-adjacent reflections from the definition of \( \sigma^m \). The Hecke relation follows from a direct check on the eigenvalues of the triangular matrix \( R|_{V \otimes V} \) from (1).

2.4. Obtaining Schur-Weyl duality. We have analogues of Theorems 1.3 and 1.5 for \( V^{\otimes m} \).

\[ \text{Theorem 2.6.} \quad \text{If } q \text{ is not a root of unity, we have:} \]

\[ \begin{align*}
&\text{(a) the images of } U_q(\mathfrak{sl}_n) \text{ and } H_q(m) \text{ in } \text{End}(V^{\otimes m}) \text{ are commutants of each other;} \\
&\text{(b) as a } H_q(m) \otimes U_q(\mathfrak{sl}_n) \text{-module, we have the decomposition} \\
&V^{\otimes m} = \bigoplus_{\lambda \vdash m, \ell(\lambda) \leq n} S_\lambda \boxtimes L_\lambda,
\end{align*} \]

where \( S_\lambda \) and \( L_\lambda \) are quantum deformations of the classical representations of \( S_m \) and \( U(\mathfrak{sl}_n) \).

Proof. We explain a proof for \( n > m \), though the result holds in general. For (a), we use a dimension count from the non-quantum case. By the definition of \( \sigma^m \) in terms of \( R \)-matrices, each algebra lies inside the commutant of the other. We now claim that \( \sigma^m(H_q(m)) \) spans \( \text{End}_{U_q(\mathfrak{sl}_n)}(V^{\otimes m}) \). If \( q \) is not a root of unity, the decomposition of \( V^{\otimes m} \) into \( U_q(\mathfrak{sl}_n) \)-isotypic components is the same as in the classical case, meaning that its commutant has the same dimension as in the classical case. Similarly, \( H_q(m) \) is isomorphic to \( \mathbb{C}[S_m] \); because \( \sigma^m \) is faithful, this means that \( \sigma^m(H_q(m)) \) has the same dimension as the classical case, and thus \( \sigma^m(H_q(m)) \) is the entire commutant of \( U_q(\mathfrak{sl}_n) \). Finally, because \( U_q(\mathfrak{sl}_n) \) is semisimple and \( V^{\otimes m} \) is finite-dimensional, \( U_q(\mathfrak{sl}_n) \) is isomorphic to its double commutant, which is the commutant of \( H_q(m) \). For (b), \( V^{\otimes m} \) decomposes into such a sum by (a), so it suffices to identify the multiplicity space of \( L_\lambda \) with \( S_\lambda \); this holds because it does under the specialization \( q \to 1 \).

\[ \text{Corollary 2.7.} \quad \text{For } n > m, \text{ the functor } \text{FS}_q : \text{Rep}(H_q(m)) \to \text{Rep}(U_q(\mathfrak{sl}_n)) \text{ defined by} \]

\[ \text{FS}_q(W) = \text{Hom}_{H_q(m)}(W, V^{\otimes m}) \]

with \( U_q(\mathfrak{sl}_n) \)-module structure induced from \( V^{\otimes m} \) is an equivalence of categories between \( \text{Rep}(H_q(m)) \) and the subcategory of weight \( m \) representations of \( U_q(\mathfrak{sl}_n) \).

Proof. From semisimplicity and the explicit decomposition of \( V^{\otimes m} \) provided by Theorem 2.6(b).
3. Yangians and degenerate affine Hecke algebras

3.1. Yang-Baxter equation with spectral parameter and Yangian. We extend the results of the previous section to the analogue of $U_q(\mathfrak{sl}_n)$ given by the solution to the Yang-Baxter equation with spectral parameter. This object is known as the Yangian $Y(\mathfrak{sl}_n)$, and it will be Schur-Weyl dual to the degenerate affine Hecke algebra $\Lambda_m$. We first introduce the Yang-Baxter equation with spectral parameter

\begin{equation}
R^{12}(u-v)R^{13}(u)R^{23}(v) = R^{23}(v)R^{13}(u)R^{12}(u-v).
\end{equation}

We may check that (2) has a solution in $\text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ given by

$$R(u) = 1 - \frac{P}{u}.$$  

This solution allows us to define the Yangian $Y(\mathfrak{gl}_n)$ via the RTT formalism.

**Definition 3.1.** The Yangian $Y(\mathfrak{gl}_n)$ is the Hopf algebra with generators $t_{ij}^{(k)}$ and defining relation

\begin{equation}
R^{12}(u-v)t_{ij}^{(k)}t_{il}^{(s)} = t_{il}^{(s)}t_{ij}^{(k)}R^{12}(u-v),
\end{equation}

where $t(u) = \sum_{i,j} t_{ij}(u) \otimes E_{ij} \in Y(\mathfrak{sl}_n) \otimes \text{End}(\mathbb{C}^n)$, $t_{ij}(u) = \delta_{ij} u^{-1} + \sum_{k \geq 1} t_{ij}^{(k)} u^{-k-1} \in Y(\mathfrak{sl}_n)[[u^{-1}]]$, the superscripts denote action in a tensor coordinate, and the relation should be interpreted in $Y(\mathfrak{sl}_n)((u^{-1}))[[u^{-1}]] \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$. The coalgebra structure is given by

$$\Delta(t_{ij}(u)) = \sum_{a=1}^n t_{ia}(u) \otimes t_{aj}(u)$$

and the antipode by $S(t(u)) = t(u)^{-1}$.

**Remark.** There is an embedding of Hopf algebras $U(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)$ given by $t_{ij} \mapsto t_{ij}^{(0)}$.

**Remark.** Relation (3) is equivalent to the relations

\begin{equation}
[t_{ij}^{(r)}, t_{kl}^{(s-1)}] - [t_{ij}^{(r-1)}, t_{kl}^{(s)}] = t_{kl}^{(r-1)} t_{ij}^{(s-1)} - t_{ij}^{(r-1)} t_{kl}^{(s-1)},
\end{equation}

for $1 \leq i,j,k,l \leq n$ and $r,s \geq 1$ (where $t_{ij}^{-1} = \delta_{ij}$). For $r = 0$ and $i = j = a$, this implies that

\begin{equation}
[t_{aa}^{(0)}, t_{kl}^{(s-1)}] = \delta_{ka} t_{al}^{(s-1)} - \delta_{al} t_{ka}^{(s-1)},
\end{equation}

meaning that $t_{ij}^{(k)}$ and $t_{ij}^{(0)}$ map between the same $U(\mathfrak{gl}_n)$-weight spaces.

**Remark.** For any $a$, the map $ev_a : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ given by

$$ev_a : t_{ij}(u) \mapsto 1 + \frac{E_{ij}}{u-a}$$

is an algebra homomorphism but not a Hopf algebra homomorphism. Pulling back $U(\mathfrak{gl}_n)$-representations through this map gives the evaluation representations of $Y(\mathfrak{gl}_n)$.

3.2. The Yangian of $\mathfrak{sl}_n$. For any formal power series $f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]$, the map

$$t(u) \mapsto f(u)t(u)$$

defines an automorphism $\mu_f$ of $Y(\mathfrak{gl}_n)$. One can check that the elements of $Y(\mathfrak{gl}_n)$ fixed under $\mu_f$ form a Hopf subalgebra.

**Definition 3.2.** The Yangian $Y(\mathfrak{sl}_n)$ of $\mathfrak{sl}_n$ is $Y(\mathfrak{sl}_n) = \{ x \in Y(\mathfrak{gl}_n) \mid \mu_f(x) = x \}$.

We may realize $Y(\mathfrak{sl}_n)$ as a quotient of $Y(\mathfrak{gl}_n)$. Define the quantum determinant of $Y(\mathfrak{gl}_n)$ by

\begin{equation}
\text{qdet} t(u) = \sum_{\sigma \in S_n} (-1)^\sigma t_{\sigma(1),1}(u) t_{\sigma(2),2}(u-1) \cdots t_{\sigma(n),n}(u-n+1)
\end{equation}

**Proposition 3.3.** We have the following:

(a) the coefficients of $\text{qdet} t(u)$ generate $Z(Y(\mathfrak{gl}_n))$;
(b) $Y(\mathfrak{gl}_n)$ admits the tensor decomposition $Z(Y(\mathfrak{gl}_n)) \otimes Y(\mathfrak{sl}_n)$;
(c) $Y(\mathfrak{sl}_n) = Y(\mathfrak{gl}_n)/\langle \text{qdet} t(u) - 1 \rangle$. 

Observe that any representation of $Y(gl_n)$ pulls back to a representation of $Y(sl_n)$ under the embedding $Y(sl_n) \to Y(gl_n)$. Further, the image of $U(sl_n)$ under the previous embedding $U(gl_n) \to Y(gl_n)$ lies in $Y(sl_n)$, so we may consider any $Y(sl_n)$-representation as a $U(sl_n)$-representation. We say that a representation of $Y(sl_n)$ is of weight $m$ if it is of weight $m$ as a representation of $U(sl_n)$.

3.3. Degenerate affine Hecke algebra. The Yangian will be Schur-Weyl dual to the degenerate affine Hecke algebra $\Lambda_m$, which may be viewed as a $q \to 1$ limit of the affine Hecke algebra.

**Definition 3.4.** The degenerate affine Hecke algebra $\Lambda_m$ is the associative algebra given by

$$\Lambda_m = \{ s_1, \ldots, s_{m-1}, x_1, \ldots, x_m \mid s_i^2 = 1, s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, [x_i, x_j] = 0, $$

$$s_ix_i - x_{i+1}s_i = 1, [s_i, s_j] = [s_i, x_j] = 0 \text{ if } |i - j| \neq 1 \}.$$

**Remark.** We have the following facts about $\Lambda_m$:

- $s_i$ and $x_i$ generate copies of $\mathbb{C}[S_m]$ and $\mathbb{C}[x_1, \ldots, x_m]$ inside $\Lambda_m$;
- the center of $\Lambda_m$ is $\mathbb{C}[x_1, \ldots, x_m]^{S_m}$;
- the elements $y_i = x_i - \sum_j < s_{ij}^m$ in $\Lambda_m$ give an alternate presentation via

$$\Lambda = \{ s_1, \ldots, s_{m-1}, y_1, \ldots, y_m \mid s_{ij} = y_1^{s_i}s_j, [y_i, y_j] = (y_i - y_j)s_{ij} \}.$$

3.4. The Drinfeld functor. We now upgrade $FS$ to a functor between $\text{Rep}(\Lambda_m)$ and $\text{Rep}(Y(sl_n))$. For a $\Lambda_m$-representation $W$, define the linear map $\rho_W : Y(gl_n) \to \text{End}(FS(W))$ by

$$\rho_W : t(u) \mapsto T^{1,*}(u - x_1)T^{2,*}(u - x_2) \cdots T^{m,*}(u - x_m),$$

where

$$T(u - x_i) = 1 + \frac{1}{u - x_i} \sum_{ab} E_{ab} \otimes E_{ab} \in \text{End}(W \otimes V \otimes V)$$

should be thought of as the image of the evaluation map $\text{ev}_a : Y(gl_n) \to U(gl_n)$ given by $t_{ij}(u) \mapsto 1 + \frac{E_{ij}}{u - a}$ at “$a = x_i$”.

**Proposition 3.5.** The map $\rho_W$ gives a representation of $Y(gl_n)$ on $FS(W)$.

**Proof.** Define $S = \sum_{ab} E_{ab} \otimes E_{ab}$. We first check the image of $\rho_W$ lies in $\text{Hom}_{S_m}(W, V^\otimes m)$. For any $f : W \to V^\otimes m$, we must check that $\rho_W(f)(s_iw) = P^{i,i+1}\rho_W(f)(w)$. Because all coefficients of $\prod_i (u - x_i)$ are central in $\Lambda_m$, it suffices to check this for

$$\tilde{\rho}_W : t(u) \mapsto \prod_i (u - x_i)\rho_W(t(u)) = \prod_i (u - x_i + S^{i,*}).$$

Notice that $(u - x_i + S^{i,*})$ commutes with the action of $s_i$ and $P^{i,i+1}$ unless $j = i, i + 1$, so it suffices to check that

$$(u - x_i + S^{i,*})(u - x_{i+1} + S^{i+1,*})f(s_iw) = P^{i,i+1}(u - x_i + S^{i,*})(u - x_{i+1} + S^{i+1,*})f(w).$$

We compute the first term as

$$(u - x_i + S^{i,*})(u - x_{i+1} + S^{i+1,*})f(s_iw)$$

$$= (u + S^{i,*})(u + S^{i+1,*})f(s_iw) - (u + S^{i+1,*})f(x_is_iw) - (u + S^{i,*})f(x_{i+1}s_iw) + f(x_ix_{i+1}s_iw).$$

Now notice that

$$(u + S^{i,*})(u + S^{i+1,*})f(s_iw) = (u + S^{i,*})(u + S^{i+1,*})P^{i,i+1}f(w)$$

$$= P^{i,i+1}(u + S^{i,*})(u + S^{i+1,*})f(w) + P^{i,i+1}[S^{i+1,*}, S^{i,*}]f(w)$$

and

$$-(u + S^{i+1,*})f(x_is_iw) = -(u + S^{i+1,*})f((s_ix_{i+1} + 1)w)$$

$$= -P^{i,i+1}(u + S^{i,*})f(x_{i+1}w) - (u + S^{i+1,*})f(w).$$
In particular, in terms of the generators $y_i$, for Theorem 3.7, an analogue of Theorem 1.5 holds for it.

Proof. We claim by induction on $k$ and $l$.

The base case $k = l = 1$ is trivial. For the inductive step, noting that $S^{l,*}S^{k+1,*} = P^{l,k+1}S^{k+1,*}$, we have

\[ (1 + \sum_{i=1}^{k} \frac{1}{u - y_i} S_i^{l,*}) (1 + \frac{S^{k+1,*}}{u - x_{k+1}}) = 1 + \sum_{i=1}^{k} \frac{1}{u - y_i} S_i^{l,*} + \frac{1}{u - x_{k+1}} \left( 1 + \sum_{i=1}^{k} \frac{1}{u - y_i} S_i^{l,*} \right) S^{k+1,*} \]

\[ = 1 + \sum_{i=1}^{k} \frac{1}{u - y_i} S_i^{l,*} + \frac{1}{u - x_{k+1}} \left( 1 + \sum_{i=1}^{k} \frac{1}{u - y_i} P^{l,k+1} \right) S^{k+1,*} \]

\[ = 1 + \sum_{i=1}^{k} \frac{1}{u - y_i} S_i^{l,*} + \frac{1}{u - x_{k+1}} \left( 1 + \sum_{i=1}^{k} P^{l,k+1} \frac{1}{u - y_i} \right) S^{k+1,*} \]

\[ = 1 + \sum_{i=1}^{k+1} \frac{1}{u - y_i} S_i^{l,*}. \]

\[ (\sum_{i=1}^{k} \frac{1}{u - y_i} S_i^{l,*} + \frac{1}{u - x_{k+1}} (1 + \sum_{i=1}^{k} P^{l,k+1} \frac{1}{u - y_i}) S^{k+1,*} - \frac{1}{u - x_{k+1}}) \]

We may check in coordinates that $[S^{l,*}, S^{k+1,*}] = [P^{l,k+1}, S^{k+1,*}]$ so that

\[ P^{l,k+1}[S^{k+1,*}, S^{l,*}] = P^{l,k+1} S^{k+1,*} S^{l,*} = S^{k+1,*} - S^{l,*}, \]

which yields the desired. To check that $\rho_W$ is a valid $Y(\mathfrak{gl}_n)$-representation, we note that the $x_i$ form a commutative subalgebra of $\Lambda_m$, hence the same proof that $ev_\alpha$ is a valid map of algebras shows that $\rho_W$ is a representation, since the action of the $x_i$ commutes with the action of $U(\mathfrak{gl}_n)$. \qed

**Lemma 3.6.** We may reformulate the action of $Y(\mathfrak{gl}_n)$ on $\text{End}(FS(W))$ via the equality

\[ \rho_W(t(u)) = 1 + \sum_{i=1}^{m} \frac{1}{u - y_i} S_i^{l,*}. \]

In particular, in terms of the generators $y_i$, we have

\[ \rho_W(t^{(k)}_{ij}) = \delta_{ij} + \sum_{i=1}^{m} y_i^k E_{ji}. \]

**Proof.** We claim by induction on $k$ that

\[ \prod_{l=1}^{k} T^{l,*}(u - x_l) = 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*}. \]

The base case $k = 1$ is trivial. For the inductive step, noting that $S^{l,*}S^{k+1,*} = P^{l,k+1}S^{k+1,*}$, we have

\[ \left( 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*} \right) (1 + \frac{S^{k+1,*}}{u - x_{k+1}}) = 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*} + \frac{1}{u - x_{k+1}} \left( 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*} \right) S^{k+1,*} \]

\[ = 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*} + \frac{1}{u - x_{k+1}} \left( 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} P^{l,k+1} \right) S^{k+1,*} \]

\[ = 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*} + \frac{1}{u - x_{k+1}} \left( 1 + \sum_{l=1}^{k} P^{l,k+1} \frac{1}{u - y_l} \right) S^{k+1,*} = 1 + \sum_{l=1}^{k+1} \frac{1}{u - y_l} S^{l,*}. \]

\[ \left( 1 + \sum_{l=1}^{k} \frac{1}{u - y_l} S^{l,*} \right) (1 + \frac{S^{k+1,*}}{u - x_{k+1}}) \]

**3.5. Schur-Weyl duality for Yangians.** The upgraded functor $FS$ is known as the Drinfeld functor, and an analogue of Theorem 1.5 holds for it.

**Theorem 3.7.** For $n > m$, the functor $FS : \text{Rep}(\Lambda_m) \to \text{Rep}(Y(\mathfrak{sl}_n))$ is an equivalence of categories onto the subcategory of $\text{Rep}(Y(\mathfrak{sl}_n))$ generated by representations of weight $m$.

**Proof.** We first show essential surjectivity. Viewing any representation $W'$ of $Y(\mathfrak{sl}_n)$ of weight $m$ as a representation of $U(\mathfrak{sl}_n)$, we have by Theorem 1.5 that $W' = FS(W)$ for some $S_m$-representation $W$. We must now extend the $S_m$-action to an action of $\Lambda_m$ by defining the action of the $y_i$. For this, we use that $W'$ is also a representation of $Y(\mathfrak{gl}_n)$ via the quotient map $Y(\mathfrak{gl}_n) \to Y(\mathfrak{sl}_n)$.

**Lemma 3.8.** We have the following:
Proof. Theorem 1.3 and reduction to isotypic components of $W$ gives (a), and (b) follows because $v$ is a cyclic vector for $U(\mathfrak{sl}_n)$ in $V^{\otimes m}$.

Define the special vectors

$$v^{(j)} = e_2 \otimes \cdots \otimes e_j \otimes e_n \otimes e_{j+1} \cdots \otimes e_m \quad \text{and} \quad w^{(j)} = e_2 \otimes \cdots \otimes e_j \otimes e_1 \otimes e_{j+1} \cdots \otimes e_m.$$  

For $w \in W$, the action of $t^{(1)}_{1n}$ on $v^{(j)} \cdot w^*$ lies in $w^{(j)} \cdot W^*$ by $U(\mathfrak{sl}_n)$-weight considerations via (5). By Lemma 3.8, we may define linear maps $\alpha_j \in \text{End}_C(W)$ by

$$t^{(1)}_{1n} (v^{(j)} \cdot w^*) = w^{(j)} \cdot \alpha_j(w)^*.$$  

Similarly, we may define maps $\beta_j, \gamma_j \in \text{End}_C(W)$ so that

$$t^{(1)}_{1n} (w^{(j)} \cdot w^*) = w^{(j)} \cdot \beta_j(w)^*$$  

and

$$t^{(2)}_{1n} (v^{(j)} \cdot w^*) = v^{(j)} \cdot \gamma_j(w)^*.$$  

Evaluate the relation $[t^{(1)}_{1n}, t^{(0)}_{11}] - [t^{(0)}_{1n}, t^{(1)}_{11}] = 0$ on $v^{(j)} \cdot w^*$ to find that $\alpha_j(w) - \beta_j(w) = 0$. Now, combining the relations

$$-[t^{(2)}_{1n}, t^{(0)}_{11}] = t^{(2)}_{1n} \quad \text{and} \quad [t^{(2)}_{1n}, t^{(0)}_{11}] - [t^{(1)}_{1n}, t^{(1)}_{11}] = t^{(1)}_{1n} - t^{(0)}_{11},$$  

we find that

$$-t^{(2)}_{1n} + [t^{(1)}_{1n}, t^{(1)}_{11}] = t^{(1)}_{1n} - t^{(0)}_{11}.$$  

Evaluating this on $v^{(j)} \cdot w^*$ implies that $-\gamma_j(w) + \alpha_j^2(w) = 0$.

Lemma 3.9. The formulas for the action of the following Yangian elements

$$t^{(1)}_{1n} = \sum_l \alpha_l E_{1n}^{(l)} \quad t^{(1)}_{11} = \sum_l \alpha_l E^{(l)}_{11} \quad t^{(2)}_{1n} = \sum_l \alpha_l^2 E_{1n}^{(l)}$$  

hold on all of $FS(W)$.

Proof. For $t_{11}^{(i)}$, because $t_{ij}^{(0)}$ commutes with $t_{11}^{(i)}$ for $i, j \notin \{1, n\}$, it suffices by Lemma 3.8(b) to verify the claim on basis vectors $v \in V$ containing $e_2, \ldots, e_{n-1}$ at most once as tensor factors. In fact, for each configuration of $e_1$’s and $e_n$’s which occur, it suffices to verify the claim for a single such basis vector. Similar claims hold for $t_{11}^{(1)}$ and basis vectors containing $e_2, \ldots, e_n$ at most once. Call basis vectors containing $r$ copies of $e_1$ and $s$ copies of $e_n$ vectors of type $(r, s)$.

The claim holds for $t_{11}^{(1)}$ for $(0, *)$ trivially and for $(1, *)$ because it holds for $w^{(j)}$. Now, we have $[t^{(1)}_{11}, t^{(0)}_{11}] = t^{(1)}_{11}$, so this implies that the claim holds for $t^{(1)}_{11}$ for $(0, *)$. Now, observe that $[t^{(1)}_{1n}, t^{(2)}_{1n}] = 0$, so replacing any $v$ of type $(r, s)$ which does not contain $e_2$ with $v'$ which has $e_2$ instead of $e_1$ in a single tensor coordinate yields

$$t^{(1)}_{11} v = t^{(1)}_{1n} t^{(1)}_{12} v' = t^{(1)}_{12} t^{(1)}_{11} v',$$  

whence the claim holds for $t^{(1)}_{11}$ on $v$ if it holds for $v'$. Induction on $r$ yields the claim for all $t^{(1)}_{11}$. Now, for $t^{(1)}_{11}$, suppose the claim holds for type $(r-1, 0)$, and choose a $v$ of type $(r, 0)$ with $e_1$ in coordinates $i_1, \ldots, i_r$, and let $v'$ be the vector containing $e_n$ instead of $e_1$ in the single tensor coordinate $i_r$. Then we have $v = t^{(1)}_{1n} v'$, so

$$t^{(1)}_{11} v = t^{(1)}_{11} t^{(0)}_{1n} v' = t^{(0)}_{1n} t^{(1)}_{11} v' + [t^{(0)}_{1n}, t^{(1)}_{11}] v' = t^{(0)}_{1n} \sum_{j=1}^{r-1} \alpha_{ij} E^{(j)}_{11} v' + \alpha_{ir} v = \left( \sum_{j=1}^{r-1} \alpha_{ij} + \alpha_{ir} \right) v,$$

which yields the claim for $t^{(1)}_{11}$ by induction on $r$. The claim for $t^{(2)}_{1n}$ follows from the relation

$$t^{(2)}_{1n} = t^{(0)}_{1n} t^{(1)}_{11} - t^{(1)}_{1n} t^{(0)}_{11} - [t^{(1)}_{1n}, t^{(1)}_{11}].$$  

\[\square\]
To conclude, we claim that the assignment \( y_i \mapsto \alpha_i \) extends the \( S_m \)-action on \( \mathcal{FS}(W) \) to a \( \Lambda_m \)-action. For this, we evaluate relations from \( Y(\mathfrak{sl}_n) \) on carefully chosen vectors in \( \mathcal{FS}(W) \). To check that \( s_i y_i = y_{i+1} s_i \), note that \( v^{(i)} \cdot w^* = v^{(i+1)} \cdot (s_i w)^* \), so acting by \( t_{1n}^{(1)} \) on both sides gives the desired
\[
 s_i w^{(i+1)} \cdot \alpha_i (w)^* = w^{(i)} \cdot \alpha_i (w)^* = w^{(i+1)} \cdot \alpha_i (s_i (w))^*.
\]
For the second relation, we evaluate
\[
-t_{1n}^{(2)} - [t_{1n}^{(1)}, t_{11}^{(1)}] = t_{1n}^{(1)} t_{11}^{(0)} - t_{11}^{(0)} t_{1n}^{(1)}
\]
on
\[
e_2 \otimes \cdots \otimes e_i \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_1 \otimes e_j \otimes \cdots \otimes e_m \cdot w^*
= e_2 \otimes \cdots \otimes e_i \otimes e_1 \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_n \otimes e_j \otimes \cdots \otimes e_m \cdot (s_{ij} w)^*,
\]
we find that
\[-(\alpha_j - \alpha_i) s_{ij} w = \alpha_i (\alpha_j (w)) - \alpha_j (\alpha_i (w)),
\]
which shows that \([\alpha_i, \alpha_j] = (\alpha_i - \alpha_j) s_{ij}\).

It remains to show that \( \mathcal{FS} \) is fully faithful. Injectivity on morphisms follows because \( \mathcal{FS} \) is fully faithful in the classical case. For surjectivity, any map \( F : \mathcal{FS}(W) \to \mathcal{FS}(W') \) of \( Y(\mathfrak{sl}_n) \)-modules is of the form \( F = \mathcal{FS}(f) \) for a map \( f : W \to W' \) of \( S_m \)-modules. Further, viewing \( W \) and \( W' \) as \( Y(\mathfrak{gl}_n) \)-modules via the quotient map, \( F \) commutes with the full \( Y(\mathfrak{gl}_n) \)-action because the center acts trivially on both \( W \) and \( W' \).

Now, because \( F \) commutes with the action of \( t_{1n}^{(1)} \), we see for all \( w \in W \) and \( v \in V^\otimes m \) that
\[
\sum_{i=1}^m E_{1n}^{(i)} v \cdot f(y_i w)^* = \sum_{i=1}^m E_{1n}^{(i)} v \cdot (y_i f(w))^*.
\]
Taking \( v = w^{(j)} \) shows that \( f(y_j w) = y_j f(w) \), so that \( f \) is a map of \( \Lambda_m \)-modules, as needed.

\[
4. \text{ QUANTUM AFFINE ALGEBRAS AND AFFINE HECKE ALGEBRAS}
\]

\[
4.1. \text{ Definition of the objects.} \quad \text{Our goal in this section will be to extend Corollary 2.7 to the case of }
\]
\( U_q(\hat{\mathfrak{sl}}_n) \) and \( \mathcal{H}_q(m) \). We first define these objects.

\[
\text{Definition 4.1. The quantum affine algebra } U_q(\hat{\mathfrak{sl}}_n) \text{ is the quantum group of the Kac-Moody algebra associated to type } A_{n-1}^{(1)}, \text{ meaning that the Cartan matrix } A \text{ is given by}
\]
\[
A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 2 \\
-1 & 0 & 0 & \cdots & -1 & 2
\end{pmatrix}.
\]

\[
\text{Remark. The obvious embedding } x_i^\pm \mapsto x_i^\pm \text{ and } q^{k_i/2} \mapsto q^{k_i/2} \text{ realizes } U_q(\hat{\mathfrak{sl}}_n) \text{ as a Hopf subalgebra of } U_q(\mathfrak{sl}_n). \text{ We say that a } U_q(\hat{\mathfrak{sl}}_n)-\text{representation is of weight } m \text{ if it is of weight } m \text{ as a } U_q(\mathfrak{sl}_n)-\text{representation.}
\]

\[
\text{Definition 4.2. The affine Hecke algebra } \mathcal{H}_q(m) \text{ is the associative algebra given by}
\]
\[
\mathcal{H}_q(m) = \langle T_1^\pm, \ldots, T_{m-1}^\pm, X_1^\pm, \ldots, X_m^\pm | [X_i, X_j] = 0, (T_i - q^{-1})(T_i + q) = 0, T_i T_{i+1} T_i = T_{i+1} T_i T_i, \quad T_i X_i T_i = q^2 X_{i+1}, [T_i, T_j] = [T_i, X_j] = 0 \text{ for } |i - j| \neq 1 \rangle.
\]
4.2. Drinfeld functor and Schur-Weyl duality. We now give an extension of Corollary 2.7 to the affine setting. The strategy is the analogue of the one we took for Yangians. For variety, we present a construction directly in the Kac-Moody presentation in this case. For a $\mathcal{H}_q(m)$-representation $W$, define the linear map

$$\rho_{q,W} : U_q(\hat{\mathfrak{sl}}_n) \to \text{End}(FS_q(W))$$

by

$$\rho_{q,W}(x_i^\pm) = \sum_{l=1}^m X_i^\pm \otimes (q^{\mp h_\theta/2})^\otimes l-1 \otimes x_\theta^\mp \otimes (q^{\mp h_\theta/2})^\otimes m-l,$$

and

$$\rho_{q,W}(q^{h_0}) = 1 \otimes (q^{-h_\theta})^\otimes m,$$

where $x_\theta^+ = E_{1n}$ and $x_\theta^- = E_{n1}$ as operators in $\text{End}(V)$, and $q^{h_\theta} = q^{h_1+\cdots+h_{n-1}}$.

**Theorem 4.3.** The map $\rho_{q,W}$ defines a representation of $U_q(\hat{\mathfrak{sl}}_n)$ on $FS_q(W)$.

**Proof.** By a direct computation of the relations of $U_q(\hat{\mathfrak{sl}}_n)$. For details, the reader may consult [CP96, Theorem 4.2]; note that the coproduct used there differs from our convention, which follows [Jim86].

**Theorem 4.4.** For $n > m$, the functor $FS_q : \text{Rep}(\mathcal{H}_q(m)) \to \text{Rep}(U_q(\hat{\mathfrak{sl}}_n))$ is an equivalence of categories onto the subcategory of $\text{Rep}(U_q(\hat{\mathfrak{sl}}_n))$ generated by representations of weight $m$.

**Proof.** The proof of essential surjectivity is analogous to that of Theorem 3.7. The action of $X_i^\pm$ is obtained by evaluation on some special basis vectors in $V^\otimes m$ and the relations of $\mathcal{H}_q(m)$ are shown to be satisfied for them from the relations of $U_q(\hat{\mathfrak{sl}}_n)$. For details, see [CP96, Sections 4.4-4.6]. The check that $FS_q$ is fully faithful is again essentially the same as in Theorem 3.7.

**References**


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