Classical Hodge theory and the Decomposition theorem via Hodge theory

1) Hodge theory and Lefschetz linear algebra

2) Semismall maps and Hard Lefschetz theorem.

3) Intersection cohomology and Decomposition theorem

1) Hodge theory and Lefschetz linear algebra

Classical Hodge theory for smooth projective complex varieties starts with the Hodge decomposition:

$$H^i(X;\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X).$$

For our purpose, we will always assume that

$$H^{p,q}(X) = 0, \quad \text{when } p \neq q.$$

The structure of interest to us is the total cohomology

$$H = \bigoplus H^i(X;\mathbb{R}).$$

We start with the axiomatized setup:

<u>Fix</u>: $H = \bigoplus H^i$: a finite dim graded $\mathbb{R}$-vector space

$\langle -,- \rangle : H \times H \to \mathbb{R}$ a symmetric, non-degenerate, graded form, $\langle H^i, H^j \rangle = 0$ if $i \neq -j$

Hence, if $b_i = \dim H^i$, then $b_i = b_{-i}$, $\forall i \in \mathbb{Z}$.

<u>Example</u>: If M is a compact manifold of dim $2n$, set $H^i := H^{i+n}(M; \mathbb{R})$. Let $\langle -,- \rangle$ be

$$\langle w_1, w_2 \rangle = \int_M w_1 \wedge w_2 .$$

If $H^{2k+1}(M; \mathbb{R}) = 0$ for any k, $\langle -,- \rangle$ is symmetric.

A <u>Lefschetz operator</u> is a map $L : H^\circ \to H^{\circ+2}$ s.t.

$$\langle Lx, y \rangle = \langle x, Ly \rangle \text{ for } \forall x, y \in H.$$

<u>Example</u>: With M as above, and $\alpha \in H^2(M; \mathbb{R})$, $\cdot \cup \alpha$ gives a Lefschetz operator.

<u>Def</u>: A Lefschetz operator L satisfies <u>the hard Lefschetz theorem (hL)</u>, if $L^i : H^{-i} \to H^i$ is an isomorphism for $\forall i$.

<u>Exercise</u>: Let $sl_2(\mathbb{R}) = \mathbb{R}\langle f, h, e\rangle$. A Lefschetz operator satisfies (hL) $\iff$ $\exists$ an action of $sl_2(\mathbb{R})$ on H s.t. $e = L$ and $hx = i \cdot x$ for $\forall x \in H^i$. Moreover, this action is unique.

<u>Example</u>: If $X \subset \mathbb{CP}^n$ is a smooth projective variety, then $L = \cup\, c_1(\mathcal{O}(1))$ satisfied (hL)

If L satisfies (hL), then we have <u>the primitive decomp</u>

$$H = \bigoplus_{i \geq 0} \underbrace{\Big(\bigoplus_{i \geq j \geq 0} L^j P_L^{-i} \Big)}_{\text{isotypic component}}, \quad \text{where } P_L^{-i} := \ker L^{i+1} \subset H^{-i}$$

sl_2 isotypic component "lowest weight" (for P_L^{-i})

$\langle -, - \rangle$ pairs H^i and H^{-i}, L^i identifies them.

Lefschetz form: $(\alpha,\beta)_L^{-i} := \langle \alpha, L^i\beta\rangle$ (symmetric)

(hL) $\Leftrightarrow$ non-degeneracy of $(-,-)_L^i \quad \forall i \geq 0$

Exercise: $(L\alpha, L\beta)_L^{-i+2} = (\alpha,\beta)_L^{-i} \qquad i \geq 2$

(hL): $H^{-i} = P_L^{-i} \oplus LP_L^{-i-2} \oplus \cdots$ is orthogonal w.r.t $(-,-)_L$

Hodge-Riemann bilinear relations: Assume $H^{odd}=0$ or $H^{even}=0$

Let min be s.t. $H^{min} \neq 0$ but $H^j = 0 \quad \forall j < min$.

$(H, \langle -,- \rangle, L)$ satisfies the Hodge-Riemann bilinear relations

(HR) if the restriction of $(-,-)_L^{min+2i}$ to P_L^{min+2i} is

$(-1)^i$-definite.

$$H^{min+2i} = L^i P_L^{min} \oplus L^{i-1} P_L^{min+2} \oplus \cdots \oplus P_L^{min+2i} \quad \text{(orthogonal)}$$

$$+ \qquad - \qquad \cdots \qquad (-1)^i \quad \Rightarrow \text{(hL)}$$

$$\Leftrightarrow \text{signature of } (-,-)_L^{min+2i} = \sum_{j\geq 0} (-1)^j \dim P_L^{min+2j}$$

Example: See first part of Page 12

2) Semismall maps and the hard Lefschetz theorem

The reference is [dCM] "The hard Lefschetz theorem and the topology of semismall maps";

In this section, we always consider a projective morphism

$$f : X \to Y$$

where X is smooth projective, and X, Y both irreducible.

Denote $Y^k := \{ y \in Y \mid \dim f^{-1}(y) = k \}$

<u>Def</u>: We say $f: X \to Y$ is <u>semismall</u>, if

$$\dim Y^k + 2k \leq \dim X =: n, \ \forall k .$$

<u>Rmk</u>: In this case, f is generically finite: $\left\{ \begin{array}{l} \text{For } k > 0 \\ \dim Y^k + k \leq n - k, \\ \text{so } f^{-1}(Y^k) \subsetneq X \end{array} \right.$

Again, let $H = \bigoplus H^i$, where $H^i := H^{n+i}(X; \mathbb{R})$.

Now we consider the Lefschetz operator:

$$L = \cup c_1(f^* A), \text{ where } A \text{ is ample on } Y.$$

Thm (dC–M): Let $f: X \to Y$, L be as above, assume that f is semismall. Then (H, L) satisfies (hL), (HR).

Example: See Page 12 – 13

To see why the semismall condition is relevant, consider a birational morphism $f: X \to Y$ between 3-folds which contracts a surface S to a point. In this case, f is not semismall. Now $L([S]) = [S] \cup f^* A \overset{\text{projection formula}}{=} 0$, so ~~the~~ (hL) doesn't hold. Completely similar method shows that (hL) of L implies f is semismall ([dCM]: Prop 2.2.7).

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dCM proof strategy:

$$\begin{matrix} (hL), (HR) & \xrightarrow[\text{Lefschetz}]{\text{weak}} & (hL) \text{ in} & \xrightarrow[\text{lemma}]{\text{limit}} & (HR) \text{ in} \\ \text{in dim } n & & \text{dim } n+1 & & \text{dim } (n+1) \end{matrix}$$

Key steps:

① Weak Lefschetz substitute (wL): Suppose $H, \langle -,- \rangle_H, L_H$,

W, $\langle -,- \rangle_W$, L_W are as above, with L_H, L_W Lefschetz operators. Suppose $\phi : H \to W$ of deg 1 s.t.

1) ϕ injective in degrees ≤ -1.
2) $\langle \alpha, L_H \beta \rangle_H = \langle \phi\alpha, \phi\beta \rangle_W$, $\phi \circ L_H = L_W \circ \phi$.
3) W satisfies (HR).

Then L_H satisfies (hL).

Pf: Fix $0 \neq h \in H^{-i}$, with $i \leq -1$, and consider $\phi(h) \in W^{-i+1}$. Then either:

1) $0 \neq L^i(\phi(h)) = \phi(L^i(h)) \Rightarrow L^i h \neq 0$, or
2) $0 = L^i(\phi(h)) \Rightarrow \phi(h) \in P_L^{-i+1} \Rightarrow$

$$0 \neq (\phi(h), \phi(h))_L^{-i+1} = \langle \phi(h), L^{i-1}\phi(h) \rangle = \langle h, L^i h \rangle \quad \square$$

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② <u>Limit lemma</u>: Suppose that $[0,\infty) \to \mathrm{Hom}(H, H(2))$, $J \mapsto L_J$ is a continuous family of Lefschetz operators satisfying (hL). If $\exists J \in (0,\infty)$ s.t. L_J satisfies (HR), then all L_J

satisfy (HR).

Pf: All L_J satisfy (hL) $\Leftrightarrow$ $(-,-)^i_{L_J}$ is a continuous family of symmetric non-degenerate forms.

Hence all have same signature. Hence all satisfy (HR). □

Sketch of Sketch of pf of Thm (dC-M):

When $n=1$, L is defined by an ample divisor on X, so it follows from classical Hodge theroy.

Assume (hL) & (HR) in dim n. In dim $n+1$,

Prop 2.1.5 in [dCM] states that ~~we can find~~ for a smooth divisor $H \in |f^*A|$, the restriction $i^*: H^*(X;\mathbb{R}) \to H^*(H;\mathbb{R})$ puts us in the situation of (wL) as in Key Step ①. Hence we have (hL) by induction.

For (HR), just note that f^*A is nef, so by Kleiman's thm, it is on the boundary of the ample cone of X, so $f^*A + \varepsilon B$ is ample, for any ~~B~~ ample B and $0 < \varepsilon \ll 1$. This puts us in the situation of limit lemma, and concludes (HR). □

<u>Warning</u>: we never introduce (HR) in general. Our definition of (HR) is only for the case $H^{p,q} = 0$ when $p \neq q$. This should be enough for our purpose.

3) Intersection cohomology and the Decomposition theorem.

To any complex variety X, we consider the <u>intersection cohomology group</u> $IH^\circ(X)$ ($\mathbb{R}$-coefficients):

(1) $IH^\circ(X)$ is a graded vector space, concentrated in degrees between 0 and $2N$, where $N = \dim_{\mathbb{C}} X$;

(2) If X is smooth, then $IH^\circ(X) = H^\circ(X)$;

(3) If X is projective, then $IH^\circ(X)$ is equipped with a non-degenerate Poincaré pairing $\langle -,- \rangle$, which is the usual Poincaré pairing for X smooth.

<u>Cautions!</u>

(1) $X \mapsto IH^\circ(X)$ is not functorial: in general, $f: X \to Y$ doesn't induce a pull-back on IH;

(2) $IH^\circ(X)$ is not a ring, but rather a module over the cohomology ring $H^\circ(X)$.

Key properties when X is projective: (BBD, Saito, dCM)

(1) multiplication by c_1 of an ample line bundle on $IH^\circ(X)$ satisfies the hard Lefschetz theorem;

(2) the groups $IH^\circ(X)$ satisfy the Hodge-Riemann bilinear relations.

According to our convention, here we consider $IH^\circ(X)[-N]$. Also (2) should be applied only to the case of pure type (p,p). We will not go through these issues though.

The main theorem on IH is the following:

Thm (Decomposition theorem) Let $f: \hat{X} \to X$ be a resolution, then $IH^\circ(X)$ is a direct summand of $H^\circ(\hat{X})$, as modules over $H^\circ(X)$. (BBD, Saito, dCM)

We will not prove this theorem, but use it to compute one example. At the end, it will be clear how it's related to section 2) in the semismall case.

Example: $Gr(2,4)$ $\dim = 4$.

Let $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4$ be the standard coordinate flag on $\mathbb{C}^4$. For $\underline{a} := \{0 = a_0 \le a_1 \le \cdots \le a_4 = 2\}$ with $a_i \le a_{i+1} \le a_i + 1$, consider

$$C_{\underline{a}} := \{V \in Gr(2,4) \mid \dim(V \cap \mathbb{C}^i) = a_i\}.$$

It's easy to see that $C_{\underline{a}} \cong \mathbb{C}^{d(\underline{a})}$, where

$$d_{\underline{a}} = 7 - \sum_{i=0}^{4} a_i .$$

This gives the cohomology table of $Gr(2,4)$

0	2	4	6	8
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}$	$\mathbb{R}$.

It can be checked the (hL) and (HR) via Schubert calculus.

Now let $X := \{V \in Gr(2,4) \mid \dim(V \cap \mathbb{C}^2) \ge 1\}$

Then $X = \overline{C_{\underline{a}}}$ where $\underline{a} = \{0,0,1,1,2\}$. Hence, X decomposes into $\{0,0,1,1,2\}^6$ $\{0,0,1,2,2\}^4$ $\{0,1,1,1,2\}^4$ $\{0,1,1,2,2\}^2$ $\{0,1,2,2,2\}^0$.

The cohomology of X are

0	2	4	6
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}$

$H^\bullet(X)$ doesn't satisfy Poincaré duality or (hL).

X has a unique singular point $V_0 = \mathbb{C}^2$. To construct a resolution of X, consider $f: \tilde{X} \to X$,

$$\tilde{X} := \{(V, W) \in Gr(2,4) \times \mathbb{P}(\mathbb{C}^2) \mid W \subset V \cap \mathbb{C}^2\}$$

and $f(V, W) = V$. Clearly f is an isomorphism over over $X \setminus \{V_0\}$, and has fiber $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ over V_0. The projection $(V, W) \mapsto W$ realizes $\tilde{X}$ as a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. This gives us the cohomology table of $\tilde{X}$:

0	2	4	6
$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}^2$	$\mathbb{R}$

Claim: $IH^\bullet(X) = H^\bullet(\tilde{X})$

Pf: Clearly the pull-back morphism $H^\circ(X) \to H^\circ(\tilde{X})$ is injective. The Decomposition theorem states that $IH^\circ(X)$ is a summand of $H^\circ(\tilde{X})$ (as $H^\circ(X)$-modules!), hence we now $IH^i(X) = H^i(\tilde{X})$ for $i \neq 2$.

Finally, we must have $IH^2(X) = H^2(\tilde{X})$, since $IH^\circ(X)$ satisfies the Poincaré duality. □

In this case, (hL) and (HR) for $IH^\circ(X)$ are equivalent to that of $H^\circ(\tilde{X})$ with $f^*\mathcal{O}_X(1)$. Note that f is semismall in our case, so this follows exactly from Thm(dCM) in Section 2).

Rmk: A large part of this note is directly taken from a lecture note and a survey of Elias and Williamson.