WEBSTER'S FUNCTORS

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1. Functors E, F: grassmanian case

We start by treating the most basic case, when the quiver Q has one vertex and no loops. Recall that in this case $\mathcal{M}^\theta(v) = T^*\operatorname{Gr}(v,w)$ when $\theta > 0$. When λ is integral, the quantization $\mathcal{A}^\theta_\lambda(v)$ is basically the sheaf of λ-twisted differential operators $D^\lambda_{\operatorname{Gr}(v,w)}$ (this is true for arbitrary λ but we have not defined the corresponding sheaf). Our goal is to introduce functors

$$F : D^b(\operatorname{Coh}(D_{\operatorname{Gr}(v,w)})) \rightleftarrows D^b(\operatorname{Coh}(D_{\operatorname{Gr}(v+1,w)})) : E$$

that will give rise to a categorical $\mathfrak{sl}_2$-action on

$$\bigoplus_v D^b(\operatorname{Coh}(D_{\operatorname{Gr}(v,w)})).$$

We start with a warm-up: we explain Nakajima's construction of the geometric $\mathfrak{sl}_2$-action on $\bigoplus_v H_{mid}(T^*\operatorname{Gr}(v,w))$.

1.1. **Warm-up: Nakajima's geometric action.** The lagrangian subvariety $\varphi^{-1}(0) \subset \mathcal{M}^\theta(v)$ is easily seen to be the zero section $\operatorname{Gr}(v,w) \subset T^*\operatorname{Gr}(v,w)$. Indeed, consider the action of $\mathbb{C}^\times$ on $\mathcal{M}^\theta(v)$ induced from the $\mathbb{C}^\times$-action on T^*R by fiberwise dilations. Under the identification of $\mathcal{M}^\theta(v)$ with $T^*\operatorname{Gr}(v,w)$, we get the action by fiberwise dilations. Note that $\varphi^{-1}(0)$ is compact and $\mathbb{C}^\times$-stable. The only lagrangian subvarierty in $T^*\operatorname{Gr}(v,w)$ with these properties is $\operatorname{Gr}(v,w)$. So we have $H_{mid}(T^*\operatorname{Gr}(v,w)) = H_{top}(\operatorname{Gr}(v,w)) = \mathbb{C}$.

We want to construct operators between $H_{top}(\operatorname{Gr}(v,w))$ and $H_{top}(\operatorname{Gr}(v+1,w))$. Note that we have a correspondence between $\operatorname{Gr}(v,w), \operatorname{Gr}(v+1,w)$: the partial flag variety $\operatorname{Fl}(v,v+1;w)$. More precisely, we have morphisms $\pi_v : \operatorname{Fl}(v,v+1,w) \twoheadrightarrow \operatorname{Gr}(v,w), \pi_{v+1} : \operatorname{Fl}(v,v+1,w) \twoheadrightarrow \operatorname{Gr}(v+1,w)$, both are locally trivial fibrations whose fibers are projective spaces $\mathbb{P}^{w-v-1}, \mathbb{P}^v$. So we have maps

$$f := \pi_{v+1*}\pi_v^* : H_{top}(\operatorname{Gr}(v,w)) \to H_{top}(\operatorname{Gr}(v+1,w)) : \pi_{v*}\pi_{v+1}^* =: e.$$

Here π_v^* is the identity map between $H_{top}(\operatorname{Gr}(v,w)) \to H_{top}(\operatorname{Fl}(v,v+1;w))$ and π_{v*} is the multiplication by the Euler characteristic of the fiber. The maps π_{v+1}^*, π_{v+1*} are defined similarly. The maps e, f equip the space $\bigoplus_v H_{top}(\operatorname{Gr}(v,w))$ with the irreducible $w+1$-dimensional representation of $\mathfrak{sl}_2$ (where $H_{top}(\operatorname{Gr}(v,w))$ has weight $w-2v$).

1.2. **Pull-back and push-forward for D-modules.** We want to define functors between derived categories of D-modules on grassmanians. The construction above suggests that the functors have to be compositions of pull-backs and push-forwards.

So let X, Y be smooth algebraic varieties and $\psi : X \to Y$ be a morphism. We are going to define the pull-back and push-forward functors between the derived categories of D-modules. For this we will need certain bimodules $D_{X\to Y}$ and $D_{X\leftarrow Y}$.

Let us define $D_{X\to Y}$ as $\psi^* D_Y$, where ψ^* is the pull-back for quasicoherent sheaves. The sheaf $\psi^\bullet D_Y$ ($\psi^\bullet$ is the naive sheaf-theoretic pullback) clearly acts by multiplications on the right. The sheaf D_X acts by multiplications on the left. Namely, $\mathcal{O}_X$ acts in a natural way. The action of Vect_X arises as follows. We can view the left D_X-module D_X as a vector bundle (of infinite rank) with a flat connection. A pull-back of a bundle with a flat connection also comes with a flat connection that gives rise to the action of the vector fields on X. Clearly, the actions of D_X and $\psi^\bullet D_Y$ commute.

Now we can set $\psi^! M := D_{X\to Y}\otimes^L_{\psi^\bullet D_Y} M[\dim X - \dim Y]$, this is a functor $D^b(\mathrm{Coh}(D_Y)) \to D^b(\mathrm{Coh}(D_X))$ (it is exact when ψ is flat). Note that this functor upgrades the usual pull-back for the quasi-coherent sheaves (up to a shift).

Let us proceed to the $\psi^\bullet D_Y$-D_X-bimodule $D_{Y\leftarrow X}$. Recall that if M is a left D_X-module, then $K_X \otimes_{\mathcal{O}_X} M$ is a right D_X-module (one can differentiate sections of $K_X \otimes_{\mathcal{O}_X} M$ using the Leibnitz rule and applying the Lie derivative to sections of K_X (up to a sign), this gives a required structure). So we can set $D_{Y\leftarrow X} := K_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{\psi^\bullet \mathcal{O}_Y} \psi^\bullet K_Y^{-1}$ and define the functor $\psi_* : D^b(\mathrm{Coh}(D_X)) \to D^b(\mathrm{Coh}(D_Y))$ by

$$\psi_* N := R\psi_\bullet(D_{Y\leftarrow X} \otimes^L_{D_X} N).$$

Below we will need two special cases:

a) ψ is a closed embedding. Then ψ_* defines an equivalence between $\mathrm{Coh}(D_X)$ and the full subcategory $\mathrm{Coh}_X(D_Y)$ in $\mathrm{Coh}(D_Y)$ consisting of all D-modules supported on X as quasi-coherent sheaves. The functor $\psi^!$ is quasi-inverse, up to a homological shift (Kashiwara's equivalence).

b) ψ is smooth and proper. In this case, ψ_* is left adjoint to $\psi^!$. Moreover, $\psi^![2(\dim Y - \dim X)]$ is left adjoint to ψ_*.

1.3. **Categorical $\mathfrak{sl}_2$-action.** We define functors

$$F : D^b(\mathrm{Coh}(D_{\mathrm{Gr}(v,w)})) \rightleftarrows D^b(\mathrm{Coh}(D_{\mathrm{Gr}(v+1,w)})) : E$$

by $F := \pi_{v+1*}\pi_v^!, E := \pi_{v*}\pi_{v+1}^!$ (we will also put some homological shifts below). These functors give a categorical $\mathfrak{sl}_2$-action in some precise sense. This basically means that the functors satisfy the defining relation of $\mathfrak{sl}_2$ on the level of K_0 (we only consider weight representations so we do not need h), and have some some natural transformations satisfying certain relations.

Here is a technically better (but equivalent way) to define the functors. Define a $D_{\mathrm{Gr}(v+1,w)}$-$D_{\mathrm{Gr}(v,w)}$-bimodule $\mathcal{F}_v$ and $D_{\mathrm{Gr}(v,w)}$-$D_{\mathrm{Gr}(v+1,w)}$-bimodule $\mathcal{E}_{v+1}$ as follows

$$\mathcal{F}_v := \iota_*(\mathcal{O}_{\mathrm{Fl}(v,v+1;w)}) \otimes_{\pi_v^\bullet(\mathcal{O}_{\mathrm{Gr}(v,w)})} \pi_v^\bullet(K_{\mathrm{Gr}(v,w)}),$$
$$\mathcal{E}_{v+1} := \iota_*(\mathcal{O}_{\mathrm{Fl}(v,v+1;w)}) \otimes_{\pi_{v+1}^\bullet(\mathcal{O}_{\mathrm{Gr}(v+1,w)})} \pi_{v+1}^\bullet(K_{\mathrm{Gr}(v+1,w)}).$$

Here ι is the embedding $\mathrm{Fl}(v, v+1; w) \hookrightarrow \mathrm{Gr}(v+1, w) \times \mathrm{Gr}(v, w)$ and we write π_v, π_{v+1} for the projections of $\mathrm{Gr}(v+1, w) \times \mathrm{Gr}(v, w)$ to the second and the first factor. Then the functors F, E are given by *convolution* (a sheaf analog of the derived tensor product). If we have three smooth varieties, X_1, X_2, X_3, then we can convolve $\mathcal{B}_{12} \in D^b(D_{X_1}\text{-}D_{X_2}\text{-bimod})$, $\mathcal{B}_{23} \in D^b(D_{X_2}\text{-}D_{X_3}\text{-bimod})$ by

$$\mathcal{B}_{12} * \mathcal{B}_{23} := R\pi_{13\bullet}\left(\pi_{12}^\bullet\mathcal{B}_{12} \otimes^L_{\pi_2^\bullet D_{X_2}} \pi_{23}^\bullet\mathcal{B}_{23}\right)[-2(\dim X_1 + \dim X_3)].$$

For example, the functor F is obtained by taking the convolution with $\mathcal{B}_{12} = \mathcal{F}_v$, where $X_1 = \mathrm{Gr}(v+1, w), X_2 = \mathrm{Gr}(v, w), X_3 = \{pt\}$.

It turns out that the objects $\mathcal{F} := \bigoplus_{v=0}^{w-1} \mathcal{F}_v$ and $\mathcal{E}$ satisfy the suitably interpreted relations of the (weight idempotent completion) of $U(\mathfrak{sl}_2)$ (saying that on the weight space of weight d we have $[e,f]\varepsilon_d = d\varepsilon_d$, where ε_d is the projection to the d-weigth space) as objects in $\bigoplus_{v,v'=0}^{w} D^b(D_{\mathrm{Gr}(v',w)}\text{-}D_{\mathrm{Gr}(v,w)}\text{-bimod})$, where the "algebra structure" is via the convolution. More precisely, we will have a relation of the sort $\mathcal{E}_{v+1} * \mathcal{F}_v \oplus ? = \mathcal{F}_{v-1} * \mathcal{E}_v \oplus ?$, where "?" stands for the sum of the several copies of the regular $D_{\mathrm{Gr}(v,w)}$-bimodule with homological shifts (and one of these ?'s is zero depending on whether $2v \geqslant v$ or $2v \leqslant w$). We will not use this in these lectures. What we will use is a connection between the convolution of D-modules and the convolution in the Borel-Moore homology, which will be explained below.

2. Functors E_k, F_k: general case

Here we are going to explain the functors

$$F_k : D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(v))) \rightleftarrows D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta)(\mathcal{A}_\lambda(v+\epsilon_k))) : E_k$$

for arbitrary Q due to Webster, [We].

2.1. Reduction in stages.

The construction of the functors E_k, F_k is based on the technique called *reduction in stages* that is classical in Symplectic geometry. This technique will also be employed for several other things.

Pick a vertex k and assume it is a sink. Also assume $\theta_k > 0$. We interpret $R(v)$ as follows. Set $\tilde{W}_k = W_k \oplus \bigoplus_{a, t(a)=k} V_{h(a)}$. So we have $R(v) = \mathrm{Hom}_{\mathbb{C}}(V_k, \tilde{W}_k) \oplus \underline{R}$, where $\underline{R}$ includes all summands of $R(v)$ that do not contain arrows from k.

Recall that $\mathcal{M}^\theta(v) = \mu^{-1}(0)^{\theta-ss}/G$, where $G = \prod_{\ell \in Q_0} G_\ell$ and $\mu = (\mu_\ell)_{\ell \in Q_0}$ with μ_ℓ being the moment map for the G_ℓ-action. Set $\underline{G} := \prod_{\ell \neq k} G_\ell, \underline{\mu} := (\mu_\ell)_{\ell \neq k}, \underline{\theta} := (\theta_\ell)_{\ell \neq k}$. Set $\mathcal{M}^{\theta_k}(v; v_k) := \mu_k^{-1}(0)^{\theta_k - ss}/G_k$. We have $\mathcal{M}^{\theta_k}(v; v_k) = T^* \mathrm{Hom}(V_k, \tilde{W}_k) ///_0^{\theta_k} G_k \times T^* \underline{R}$ because G_k acts on $T^*\underline{R}$ trivially. But as we have seen above $T^* \mathrm{Hom}(V_k, \tilde{W}_k) ///_0^{\theta_k} G_k = T^* \mathrm{Gr}(v_k, \tilde{W}_k) \times T^* \underline{R}$. So

$$\mathcal{M}^{\theta_k}(v; v_k) = T^* \mathrm{Gr}(v_k, \tilde{W}_k) \times T^* \underline{R}$$

This variety still comes with a natural Hamiltonian $\underline{G}$-action whose moment map is $\underline{\mu}$.

A key point in the reduction in stages is the following formula

$$\mu^{-1}(0)^{\theta-ss}/G = (\mu^{-1}(0)^{\theta-ss}/G_k)/\underline{G} = \underline{\mu}^{-1}(0)^{\underline{\theta}-ss}/\underline{G},$$

where $\mathcal{M}^{\theta_k}(v; v_k)^{\underline{\theta}-ss}$ can be defined as the image of $\mu_k^{-1}(0)^{\theta-ss} \subset \mu_k^{-1}(0)^{\theta_k - ss}$ (it also can be defined using GIT: we have the notion of stability for reductive group actions on projective (or projective over affine) schemes equipped with an ample equivariant line bundle; we are not going to elaborate on this). In any case, note that $(T^*R)^{\theta-ss} \subset (T^*R)^{\theta_k - ss}$, this is because a $(G, n\theta)$-semiinvariant polynomial is automatically $(G_k, n\theta_k)$-semiinvariant. In words, to reduce the G-action, we first reduce the G_k-action and then reduce the $\underline{G}$-action.

2.2. Reduction in stages on the level of D-modules.

We are going to use the reduction in stages strategy in order to produce the functors E_k, F_k.

Consider the reduction $\mathcal{A}_{\lambda_k}^{\theta_k}(v; v_k) = D_R ///_{\lambda_k}^{\theta_k} G_k$. Similarly to the above,

$$\mathcal{A}_{\lambda_k}^{\theta_k}(v; v_k) = D_{\mathrm{Gr}(v_k, \tilde{W}_k)}^{\lambda_k} \otimes D_{\underline{R}}.$$

On this sheaf we have a natural diagonal action of $\underline{G}$ that comes with a quantum comoment map. Then we still get the reduction in stages realization

$$\mathcal{A}^\theta_\lambda(v) = \mathcal{A}^{\theta_k}_{\lambda_k}(v; v_k)///^{\underline{\theta}}_{\underline{\lambda}}\underline{G}.$$

We will need a consequence of this on the level of the categories of modules. Recall, Lemma 1.6 in Lecture 3, that $\mathrm{Coh}(\mathcal{A}^\theta_\lambda(v))$ is the quotient of $D(R)\text{-mod}^{G,\lambda}$ by the Serre subcategory $D(R)\text{-mod}^{G,\lambda}_{\theta-uns}$ of all modules with θ-unstable support. Let $\pi^\theta(v)$ denote the quotient functor. We are going to factorize the functor $\pi^\theta(v)$, the intermediate category will be $\mathrm{Coh}(\mathcal{A}^{\theta_k}_{\lambda_k}(v; v_k))^{\underline{G},\underline{\lambda}}$ (where the superscript means the category of $(\underline{G},\underline{\lambda})$-equivariant objects). This category is the quotient of $D(R)\text{-mod}^{G,\lambda}$ by the Serre subcategory $D(R)\text{-mod}^{G,\lambda}_{\theta_k-uns}$ of objects whose support is θ_k-unstable for the action of G_k, let $\pi^{\theta_k}(v)$ denote the quotient functor. Then we have $\pi^\theta(v) = \pi^{\underline{\theta}}(v) \circ \pi^{\theta_k}(v)$, where $\pi^{\underline{\theta}}(v) : \mathrm{Coh}^{\underline{G},\underline{\lambda}}(\mathcal{A}^{\theta_k}_{\lambda_k}(v; v_k)) \twoheadrightarrow \mathrm{Coh}(\mathcal{A}^\theta_\lambda(v))$ is also a Hamiltonian reduction functor.

2.3. **Nakajima's construction via reduction in stages.** Here we are going to recall the construction of operators f_k, e_k on $\bigoplus_v H_{mid}(\mathcal{M}^\theta(v))$. Essentially, this is due to Nakajima, [Nak1, Section 10].

We are going to assume that $\theta_\ell > 0$ for all ℓ. In the construction we are going to assume that k is a sink in Q (we can achieve that by changing the orientation of Q, recall that this does not affect the quiver varieties). We start by defining the correspondence $Z \subset \mathcal{M}^\theta(v) \times \mathcal{M}^\theta(v + \epsilon_k)$, where ϵ_k denotes the coordinate vector. The subvariety Z will be largangian for the symplectic form $\pi_2^*\omega_2 - \pi_1^*\omega_1$ on $\mathcal{M}^\theta(v) \times \mathcal{M}^\theta(v + \epsilon_k)$, where ω_1, ω_2 are the symplectic forms on $\mathcal{M}^\theta(v), \mathcal{M}^\theta(v + \epsilon_k)$, and π_1, π_2 are the projections.

First let us treat the single vertex case. Let $Z_k \subset T^*\mathrm{Gr}(v_k, \tilde{W}_k) \times T^*\mathrm{Gr}(v_k + 1, \tilde{W}_k)$ be the conormal bundle to $\mathrm{Fl}(v_k, v_k + 1, \tilde{W}_k) \subset \mathrm{Gr}(v_k, \tilde{W}_k) \times \mathrm{Gr}(v_k + 1, \tilde{W}_k)$. Note that Z_k has the following important property, the moment maps for the two $\mathrm{GL}(\tilde{W}_k)$-actions on $T^*\mathrm{Gr}(v_k, \tilde{W}_k) \times T^*\mathrm{Gr}(v_k + 1, \tilde{W}_k)$ (on each of the two factors) coincide. So we have a well-defined map $\underline{\mu}_Z : Z_k \times T^*\underline{R} \to \underline{\mathfrak{g}}$.

Let us write $\underline{\mu}_v, \underline{\mu}_{v+\epsilon_k}$ for the moment maps of the $\underline{G}$-actions on $\mathcal{M}^{\theta_k}(v; v_k), \mathcal{M}^{\theta_k}(v + \epsilon_k, v_k + 1)$. We get $\underline{G}$-equivariant morphisms $\pi_{v;v_k} : \underline{\mu}_Z^{-1}(0) \to \underline{\mu}_v^{-1}(0)$ and $\pi_{v+\epsilon_k;v_k+1} : \underline{\mu}_Z^{-1}(0) \to \underline{\mu}_{v+\epsilon_k}^{-1}(0)$. Now comes a part, where the condition on θ (recall that we have assumed that $\theta_\ell > 0$ for all ℓ) becomes important.

Lemma 2.1 (Nakajima). *We have* $\pi^{-1}_{v,v_k}(\underline{\mu}_v^{-1}(0)^{\underline{\theta}-ss}) = \pi^{-1}_{v+\epsilon_k,v_k+1}(\underline{\mu}^{-1}_{v+\epsilon_k}(0)^{\underline{\theta}-ss})$.

Let $Z_k^{\underline{\theta}-ss}$ denote one of the two equal pre-images. The group $\underline{G}$ acts on $Z^{\underline{\theta}-ss}$ freely because it acts freely on $\underline{\mu}_v^{-1}(0)^{\underline{\theta}-ss}$, for example. So we get morphisms of quotients

$$Z := Z_k^{\underline{\theta}-ss}/\underline{G} \to \mathcal{M}^\theta(v), \mathcal{M}^\theta(v + \epsilon_k).$$

Let us denote these morphisms by $\pi_v, \pi_{v+\epsilon_k}$. They give rise to an embedding $Z \hookrightarrow \mathcal{M}^\theta(v) \times \mathcal{M}^\theta(v + \epsilon_k)$.

The operators f_k, e_k can be defined using the convolution in the Borel-Moore homology. It is the homology theory that works well for non-compact algebraic varieties (such as cotangent bundles). In particular, every irreducible algebraic variety X has the fundamental class in $H^{BM}_{2\dim_{\mathbb{C}} X}(X)$ to be denoted by $[X]$. Moreover, if X is an equi-dimensional variety, $H^{BM}_{2\dim_{\mathbb{C}} X}(X)$ has a basis of (the fundamental classes of) the irreducible components. More on Borel-Moore homology can be found in [CG, Section 2].

The convolution is defined similarly to what was done for D-modules (the tensor product becomes the intersection). To get the operators e_k, f_k between the homology of $\mathcal{M}^\theta(v), \mathcal{M}^\theta(v+\epsilon_k)$ we convolve with (the fundamental class of) Z.

Let us provide some more details on convolution. We need an operator $H_*(\varphi_v^{-1}(0)) \to H_*(\varphi_{v+\epsilon_k}^{-1}(0))$ that preserves the top parts. Note that that the variety $\varphi_?^{-1}(0)$ is compact so the Borel-Moore homology is the same as the usual homology. The general construction is as follows. Let Y_1, Y_2 be two smooth varieties, we assume that Y_2 is equi-dimensional of complex dimension d. Let $Z_2 \subset Y_2, Z_{12} \subset Y_1 \times Y_2$ be two subvarieties such that the projection $\pi_1 : Z_{12} \cap (Y_1 \times Z_2) \to Y_1$ is proper, let Z_1 denote the image. Then we have a convolution map $H_i^{BM}(Z_{12}) \times H_j^{BM}(Z_2) \to H_{i+j-2d}^{BM}(Z_1), (c_{12}, c_2) \mapsto c_{12} * c_2 := \pi_{1*}(c_{12} \cap Y_1 \boxtimes c_2)$. Here π_2 is the projection $Y_1 \times Z_2 \to Z_2$.

Let us get back to our situation. The following is due to Nakajima.

Lemma 2.2. *The image of the map $Z \cap (\mathcal{M}^\theta(v+\epsilon_k) \times \varphi_v^{-1}(0)) \to \mathcal{M}^\theta(v+\epsilon_k)$ (which is, obviously, proper) is contained in $\varphi_{v+\epsilon_k}^{-1}(0)$. A similar claim with v and $v+\epsilon_k$ swapped is also true.*

After doing a dimension count we see that the convolution with Z defines maps between $H_{top-j}(\varphi_v^{-1}(0)) \rightleftarrows H_{top-j}(\varphi_{v+\epsilon_k}^{-1}(0))$. These are the operators f_k, e_k we need.

2.4. **Convolution vs supports and characteristic cycles.** In order to relate the functors $F_k : D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(v))) \rightleftarrows D^b(\mathrm{Coh}(\mathcal{A}_\lambda^\theta(v+\epsilon_k))) : E_k$ (still to be produced) to the Nakajima operators, we will need to understand the compatibility of the D-module convolution with supports and characteristic cycles.

The following claim is a direct consequence of the definition of convolution.

Lemma 2.3. *Let $\mathcal{B} \in D_{X_1}$ - D_{X_2}*-bimod *and $M \in \mathrm{Coh}(D_{X_2})$. Then all homology of $\mathcal{B} * M$ are supported on*

$$\overline{\pi_1\left(\operatorname{Supp}\mathcal{B} \cap \pi_2^{-1}(\operatorname{Supp} M)\right)}$$

Let us proceed to the characteristic cycles. Assume, for simplicity, that $\mathcal{B}$ is holonomic, i.e., the support is lagrangian. Then we have the following lemma.

Lemma 2.4. *Suppose that Z_1, Z_2 are in the discussion of the convolution are lagrangian. Further, assume that the restriction of π_1 to $\operatorname{Supp}\mathcal{B} \cap \pi_2^{-1}(\operatorname{Supp} M)$ is proper. Then the characteristic cycle of $\mathcal{B} * M$ (i.e., $\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{CC}(H_i(\mathcal{B} * M))$) equals to $\operatorname{CC}(\mathcal{B}) * \operatorname{CC}(M)$.*

2.5. **Functors E_k, F_k.** We have functors

$$F : D^b(\mathrm{Coh}(\mathcal{A}_0^{\theta_k}(v; v_k))) \rightleftarrows D^b(\mathrm{Coh}(\mathcal{A}_0^{\theta_k}(v+\epsilon_k; v_k+1))) : E$$

given by convolving with $\mathcal{F} \boxtimes D_{\underline{R}}, \mathcal{E} \boxtimes D_{\underline{R}}$. This can be extended to arbitrary $\lambda_k \in \mathbb{Z}$ (by twisting with the corresponding line bundles on grassmanians).

Note that the bimodules $\mathcal{F}, \mathcal{E}$ are $\underline{G}$-equivariant (at least, informally, this should be clear). So the convolutions with $\mathcal{F} \boxtimes D_{\underline{R}}, \mathcal{E} \boxtimes D_{\underline{R}}$ define functors between *equivariant derived categories*

$$F : D^b_{\underline{G},\underline{\lambda}}(\mathrm{Coh}(\mathcal{A}_0^{\theta_k}(v; v_k))) \rightleftarrows D^b_{\underline{G},\underline{\lambda}}(\mathrm{Coh}(\mathcal{A}_0^{\theta_k}(v+\epsilon_k; v_k+1))) : E.$$

These are triangulated categories with t-structures whose hearts are the usual abelian categories of twisted equivariant modules. We have a natural functor

$$D^b(\mathrm{Coh}^{\underline{G},\underline{\lambda}}(\mathcal{A}_0^{\theta_k}(v; v_k))) \to D^b_{\underline{G},\underline{\lambda}}(\mathrm{Coh}(\mathcal{A}_0^{\theta_k}(v; v_k)))$$

that is an equivalence if the moment map for $\underline{G}$ is flat. This map is not flat in general, but we do not care, roughly speaking, because we are only interested in the stable locus, where the action is free (and hence the moment map is free). In particular,

$$D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda(v))) = D^b_{\underline{G},\underline{\lambda}}(\mathrm{Coh}(\mathcal{A}^{\theta_k}_0(v;v_k)))/D^b_{\underline{G},\underline{\lambda}}(\mathrm{Coh}(\mathcal{A}^{\theta_k}_0(v;v_k)))_{\theta-uns},$$

where we mod out all complexes whose homology has θ-unstable support.

Applying Lemma 2.1 and Lemma 2.3, we see that the functors E, F between the equivariant derived categories preserve the subcategories with unstable support. So they induce the functors on the quotient categories

$$F_k : D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda(v))) \rightleftarrows D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda(v+\epsilon_k))) : E_k.$$

Further, applying Lemma 2.3 now with Lemma 2.2, we see that the functors preserve the subcategories $D^b_{\varphi_?^{-1}(0)}(\cdot)$.

Let us see why these functors induce the Nakajima operators f_k, e_k on the level of characteristic cycles. Note that the characteristic cycles of both $\mathcal{F}$ and $\mathcal{E}$ coincide with (the fundamental class) of the conormal bundle to $\mathrm{Fl}(v_k, v_k+1; \tilde{w}_k)$. So the Nakajima operators involve convolution with the characterestic cycles of the bimodules used to define the functors F_k, E_k (after passing to the quotient by $\underline{G}$ but taking the quotient does not change anything in this case). So our claim follows essentially from Lemma 2.4 ("essentially" has to do with the fact that the conditions of the lemma are not quite satisfied but since we only care about the θ-stable locus, this doesn't matter).

3. Functors E_α, F_α

An arbitrary real root in a Kac-Moody algebra is obtained from a simple one by conjugating with a Weyl group element. It turns out that there is a Weyl group action by isomorphisms on quiver varieties (changing the dimension vector and the stability condition) and on their quantizations (changing also the parameter). The functors E_α, F_α will be obtained from E_i, F_i by twisting with these isomorphisms (and composing with derived equivalences corresponding to switching the stability conditions though this can be – and should be – omitted).

3.1. **Weyl group action: classical case.** Recall that we assume that Q has no loops. The Weyl group $W(Q)$ is generated by elements $s_k, k \in Q_0$, subject to the usual relations. Let us define a linear action of $W(Q)$ on $\mathbb{C}^{Q_0}$ (the space where θ's and λ's live) and an affine action of $W(Q)$ on $(\mathbb{C}^{Q_0})^*$ (the space where v's live).

Define the linear automorphism s_k of $\mathbb{C}^{Q_0}$ by $(s_k\lambda)_k = -\lambda_k, (s_k\lambda)_\ell = \lambda_\ell + n_{k\ell}\lambda_k$. Here $n_{k\ell}$ is the number of arrows between k and ℓ (in both directions). This defines a linear action of the group $W(Q)$.

We also have an affine automorphism of $(\mathbb{C}^{Q_0})^*$ given by $(s_k \cdot v)_k := w_k + \sum_{t(a)=k} v_{h(a)} + \sum_{h(a)=k} v_{t(a)} - v_k, (s_k v)_\ell = v_\ell$, the underlying linear automorphism is dual to the automorphism s_k of $\mathbb{C}^{Q_0}$. Again, this defines an affine action of $W(Q)$. The point of this action is as follows. Recall that v defines a weight ν for $\mathfrak{g}(Q)$. The weight corresponding to $s_k \cdot v$ is $s_k\nu$.

The following important proposition is due to Maffei, [Ma], (similar results were also obtained by Lusztig and Nakajima).

Proposition 3.1. *We have a symplectomorphism $\mathcal{M}^\theta_\lambda(v) \cong \mathcal{M}^{s_k\theta}_{s_k\lambda}(s_k \cdot v)$ provided $\lambda_k \neq 0$ or $\theta_k \neq 0$. The bundle $\mathcal{O}(\chi)$ on $\mathcal{M}^\theta_\lambda(v)$ corresponds to the bundle $\mathcal{O}(s_k\chi)$ on $\mathcal{M}^{s_k\theta}_{s_k\lambda}(s_k \cdot v)$.*

The last claim in the proposition shows that the following diagram commutes.

$$\begin{tikzcd} \mathbb{C}^{Q_0} \arrow[r] \arrow[d, "s_k"] & H^2_{DR}(\mathcal{M}^\theta_\lambda(v)) \arrow[d, "s_{k*}"] \\ \mathbb{C}^{Q_0} \arrow[r] & H^2_{DR}(\mathcal{M}^{s_k\theta}_{s_k\lambda}(s_k \cdot v)) \end{tikzcd}$$

Let us briefly explain how the isomorphism s_k is constructed. First, let us consider the case when Q has a single vertex. We already know the claim of the theorem when $\lambda = 0$ (two realizations of $T^*\operatorname{Gr}(?, \bullet)$ explained in the first lecture). Extending this to arbitrary $\lambda \in \mathbb{C}$ is not difficult. Then one extends to the general case using reduction in stages. The claim that the moment maps transform as claimed (the level λ corresponds to the level $s_k\lambda$) is classical (compare with reflection functors for the representations of deformed preprojective algebras) and the claim about the stability conditions is harder.

Now let $\sigma = s_{i_k} \dots s_{i_1} \in W(Q)$. We can vary θ inside its chamber so that $\theta_{i_1} \neq 0, (s_{i_1}\theta)_{i_2} \neq 0$, etc. So we get an isomorphism

$$\mathcal{M}^\theta_\lambda(v) \cong \mathcal{M}^{\sigma\theta}_{\sigma\lambda}(\sigma \cdot v) \tag{1}$$

3.2. **Weyl group action: quantum case.** Now we are going to produce a quantum version of (1).

Proposition 3.2. *Under the isomorphism $\mathcal{M}^\theta(v) \cong \mathcal{M}^{\sigma\theta}(\sigma \cdot v)$, the sheaf of filtered algebras $\mathcal{A}^{sym,\theta}_\lambda(v)$ (with respect to the filtration induced by the dilation action) on the left hand side becomes $\mathcal{A}^{sym,\sigma\theta}_{\sigma\lambda}(\sigma \cdot v)$.*

Proof. From the commutative diagram above, one deduces that the two quantizations have the same period and we are done by the classification results from Lecture 2 (Section 2.3 there). ☐

It is also possible (and useful) to arrange the proof using reduction in stages.

3.3. **Construction of the functors E_α, F_α.** Now let us pick a real root α for the subalgebra $\mathfrak{a}$ meaning that $\alpha \cdot \lambda \in \mathbb{Z}$. Since α is a real root, we see that there is $\sigma \in W(Q)$ with $\sigma\alpha = \alpha^i$. Recall that $\mathcal{A}^\theta_\lambda(v) = \mathcal{A}^{sym,\theta}_{\lambda+\rho(v)}(v)$, where we write $\rho(v)$ for the element of $\mathbb{C}^{Q_0}$ corresponding to the character $-\frac{1}{2}\operatorname{tr}_R$. So $\mathcal{A}^\theta_\lambda(v) \cong \mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma \cdot v)$, where λ' is determined from $\lambda' + \rho(\sigma v) = \sigma(\lambda + \rho(v))$. One can show that $\lambda'_i = \lambda' \cdot \alpha^i \in \mathbb{Z}$ (because $\sigma\rho(v) - \rho(\sigma \cdot v) \in \mathbb{Z}^{Q_0}$). So we get functors

$$F_i^+ : D^b(\operatorname{Coh}(\mathcal{A}^{\theta+}_{\lambda'}(\sigma \cdot v))) \rightleftarrows D^b(\operatorname{Coh}(\mathcal{A}^{\theta+}_{\lambda'}(\sigma \cdot v + \alpha^i))) : E_i^+$$

Here we write θ^+ for the stability condition with all positive components. From Proposition 2.3 of Lecture 3 we see that the bounded derived categories corresponding to different stability conditions are equivalent. Using these equivalences we can pull functors F_i^+, E_i^+ to

$$F_i : D^b(\operatorname{Coh}(\mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma \cdot v))) \rightleftarrows D^b(\operatorname{Coh}(\mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma \cdot v + \alpha^i))) : E_i.$$

Then we set $F_\alpha := \sigma_*^{-1} \circ F_i \circ \sigma_*, E_\alpha := \sigma_*^{-1} \circ E_i \circ \sigma_*$, where we write σ_* for the push-forward functor corresponding to the isomorphism σ.

We need two properties from the functors F_α, E_α:

(1) F_α, E_α preserve the subcategories $D^b_{\varphi^{-1}(0)}(\cdot)$,

(2) and give maps f_α, e_α (up to a sign) after passing to the characteristic cycles.

(1) follows from the construction: all functors involved in the construction preserve complexes whose homology are supported on $\varphi_?^{-1}(0)$. (2) is much more tricky. In checking that two nontrivial steps are involved. First, we need to check that the derived equivalences corresponding to changing the stability conditions do not change the characteristic cycles (we have mentioned this property already in the previous lecture). The second thing is that, up to a sign, σ_* acts on H_{mid}'s in the same way as the operator coming from the representation of $\mathfrak{g}(Q)$. Both claims involve studying wall-crossing functors to be introduced in the next lecture.

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