About the construction of the moduli space of semistable sheaves

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Abstract

Following [3] very closely, we construct the moduli space of semistable sheaves on a projective scheme over an algebraically closed field of characteristic zero. We begin by explaining why such a space is open, bounded, and proper. Next, we show that this space is actually a projective scheme, constructing it as a quotient of a certain Quot scheme, using geometric invariant theory.

1 Introduction

The name of the seminar is "Moduli of sheaves on $K3$ surfaces", so before we actually talk about these objects and study their geometry, we need to construct them. Even if the title refers to $K3$ surfaces only, we can actually construct such a moduli of (semistable) sheaves on any projective scheme, by which we mean a space whose closed points correspond roughly to semistable sheaves. Semistable sheaves appear naturally in the classification problem of vector bundles over an algebraic variety − the moduli space of vector bundles is usually not proper, so we have to throw in some other sheaves, close to vector bundles, which help us compactify this space. These objects will be semistable sheaves which are not locally free.

However, even if one is interested in parametrizing sheaves only on a projective variety, one will be forced to restrict to a small class of sheaves in order to construct a reasonable space. Semistable sheaves are such a class of sheaves, and every sheaf is related to semistable sheaves via the Harder-Narasimhan filtration. Thus, the existence of such a space can be seen as a method of classifying sheaves on a given projective scheme, so it certainly has an intrinsic purpose.

One is led to study these spaces for $K3$ surfaces because of their spectacular properties: two dimensional moduli spaces of sheaves on a $K3$ surface are again $K3$ surfaces, not necessarily isomorphic to the initial $K3$, but with equivalent derived categories. Also, higher dimensional moduli spaces give examples of irreducible symplectic (or even hyperkahler) manifolds. In these notes, we only discuss the first question raised in this paragraph, namely what is the moduli of (semistable) sheaves on a projective variety and how it can be constructed.

A geometric invariant theory construction shows that this space is actually a projective scheme. However, we ignore this construction for the first part of the notes and we study geometric properties of the moduli space without referring to the GIT construction. In this part of the notes, the nature of the space will not matter as all the properties can be formulated in terms of the moduli functor we will want to corepresent, or, equivalently, in terms of flat families of semistable sheaves. The reason is that the arguments used to establishing openness, boundedness, and properness of the space can be employed in understanding other moduli spaces (for example, certain moduli spaces of complexes of sheaves) where there is no GIT construction available to construct the space as a scheme. Of course, boundedness has to be established a priori of the GIT construction anyway, in order to realize semistable sheaves as points of a finite type scheme. Thus, we will give two proofs of properness, one via Langton's criterion, and one following from the GIT construction.

The plan for the lecture/ the notes is the following: we begin in section 2 by reviewing some definitions and results we will need later in the notes, such as the definition of semistable sheaves and Kleiman's boundedness criterion, and by explaining what we mean by the moduli space of sheaves of a scheme. We also show that the family of semistable sheaves on a smooth projective curve is bounded. In section 3 we prove openness for the moduli space of interest. In section 4 we discuss boundedness− the most important ingredients are the Grauert- Mulich theorem and the Le Potier- Simpson estimates. The case of higher dimensional varieties can be reduced to the case of curves via these results and Kleiman's criterion. In section 5 we prove Langton's theorem [6], which shows properness of the moduli of stable sheaves and gives a replacement for properness for semistable sheaves. In section 6 we explain how semistable sheaves can be seen as invariant points of a certain Quot scheme under the action of $GL(V)$, and we set up the GIT construction. Finally, in section 7, we prove the technical identification of (semi)stable sheaves and (semi)stable points in the GIT sense for our particular case, and conclude that the moduli space is actually a projective scheme.

Before starting, we should mention that the first construction of the moduli space of semistable sheaves was given by Gieseker and Maruyama, and that the proof we present is due to Simpson.

2 Semistable sheaves and bounded families of sheaves

In these notes, k will be always an algebraically closed field of characteristic zero, X will denote a projective scheme over k, and $\mathcal{O}_X(1)$ will be an ample line bundle on X.

We need the characteristic zero assumption because we will use the Grauert-Mulich theorem to establish boundedness for the moduli space. However, boundedness can be established over fields of characteristic p as well [4] and this result can be used to construct a moduli space of semistable sheaves in this setting using the methods of the present article [5].

As we said in the introduction, we begin by recalling what a (Gieseker) semistable sheaf is. For X and $\mathcal{O}(1)$ as above, we define the Hilbert polynomial $P(E,t) := \chi(E \otimes \mathcal{O}(t))$ for any coherent sheaf E on X. The leading coefficient of P_E is $a_d = \frac{r}{d!}$, where d is the dimension of the support of E and r is the multiplicity of the sheaf E. We further define the reduced Hilbert polynomial $p(E,t) = \frac{P(E,t)}{a_d}$. A coherent sheaf E of dimension d is called (Gieseker) (semi)stable if E is pure and for any proper subsheaf $F \subset E$ we have $p(F)(\leq) < p(E)$. We also define the slope $\mu'(E) := \frac{a_{d-1}}{a_d}$, where a_i are the coefficients of the Hilbert polynomial.

We should comment a little on the role and definition of μ'. Recall from the first lecture that, besides Gieseker stability, we have also discussed $\mu-$stability, which had, however, in the definition the requirement that the sheaf is supported on the full scheme X. In this case, we have defined

$$\mu(E) = \frac{\deg(E)}{\operatorname{rank}(E)}.$$

We can define similar stability conditions for more general classes of sheaves on X as follows. Define $\mathrm{Coh}_d(X)$ to be the subcategory of $\mathrm{Coh}(X)$ consisting of sheaves of dimension $\leq d$; also, define $\mathrm{Coh}_{d,e}(X)$ to be the quotient category $\mathrm{Coh}_d(X)/\mathrm{Coh}_{e-1}(X)$. Thus, the objects of $\mathrm{Coh}_{d,e}(X)$ are the same as the objects of $\mathrm{Coh}_d(X)$, and morphism $F \to G$ are equivalence classes of diagrams $F \leftarrow F' \to G$ such that F, F', and G are in $\mathrm{Coh}_d(X)$, both maps are in $\mathrm{Coh}_d(X)$, and such that $F' \to F$ has both the kernel and the cokernel supported in dimension $\leq e-1$. Similarly, we define $\mathbb{Q}[T]_d = \{P \in \mathbb{Q}[T] | \deg(P) \leq d\}$ and $\mathbb{Q}[T]_{d,e} = \mathbb{Q}[T]_d/\mathbb{Q}[T]_{e-1}$. We will have a well-defined Hilbert polynomial map

$$P_{d,e} : \mathrm{Coh}_{d,e} \to \mathbb{Q}[T]_{d,e}.$$

Let $T_{d-1}(E)$ be the maximal subsheaf of E whose support is in dimension $d-1$ or lower. We call $E \in \mathrm{Coh}_{d,e}$ pure if $T_{d-1}(E) \in \mathrm{Coh}_{d,e}$ is zero, i.e.

E is pure in dimensions e and higher. Finally, we define $E \in \mathrm{Coh}_{d,e}$ to be (semi)stable if E is pure in $\mathrm{Coh}_{d,e}$ and if for all non-trivial proper subsheaves F,

$$p_{d,e}(F) \leq p_{d,e}(G).$$

This notion is a generalization of both Gieseker stability and $\mu-$stability. For $d = \dim(X)$ and $e = d - 1$ we recover $\mu-$stability. Thus, if we want to define $\mu-$stability for sheaves not necessarily supported in full dimension, we can do it using this new notion: a coherent sheaf E of dimension d is called $\mu-$(semi)stable if it is (semi)stable in $\mathrm{Coh}_{d,d-1}$, condition which can be rephrased in function of μ' only: E is $\mu-$(semi)stable if and only if $T_{d-1}(E) = T_{d-2}(E)$ and $\mu'(F)(\leq) < \mu'(E)$ for all proper subsheaves $F \subset E$. For sheaves E whose support is X, we have

$$\mu(E) = a_d(\mathcal{O}_X)\mu'(E) - a_{d-1}(\mathcal{O}_X).$$

Now, we would like to define the moduli space of semistable sheaves. For this, fix a polynomial $P \in \mathbb{Q}[X]$. Define a functor $\Phi : \mathrm{Sch/k}^{op} \to \mathrm{Sets}$, which will be the functor which we want to corepresent, by $\Phi(S)$ is the set of isomorphism classes of $S-$flat families of semistable sheaves on X with Hilbert polynomial P up to equivalence, where we say that two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there exists a line bundle L on S such that $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$. We would like to find a scheme that represent this functor and call it the moduli space of semistable sheaves.

However, this functor is not representable by a scheme in general. Indeed, if there exist semistable sheaves F_1 and F_2 with $\mathrm{Ext}^1(F_2, F_1) \neq 0$, choose F a non-trivial extension. We can construct a flat family $\mathcal{F}$ of semistable sheaves on $\mathbb{A}^1$ such that $\mathcal{F}_0 \cong F_1 \oplus F_2$ and $\mathcal{F}_t \cong F$ by taking the line in $\mathrm{Ext}^1(F_2, F_1)$ corresponding to F. We also have the constant $F_1 \oplus F_2$ family on $\mathbb{A}^1$. Now, the map from $\mathbb{A}^1 \to M$, where M is the potential moduli scheme has to be constant in both cases, because it is constant when restricted to $\mathbb{A}^1 - 0$. However, the two families are different, and so their "defining" maps to M should be different. There are two lessons we learn from here: first, we should look for a coarse moduli space instead, that is, to a scheme M with a natural transformation $\Phi \to \mathrm{Hom}(-, M)$ which is universal with respect to all these natural transformations. Second, we should identify semistable sheaves with the same Harder-Narasimhan factors and try to see whether we can find a moduli space which sees these S-equivalence classes instead. Following [3], if we can find a coarse moduli space M for Φ, we say that Φ is **corepresentable** by M.

As mentioned in the introduction, we are vague about the nature of the moduli space and call it simply a "space". It will not matter in the beginning if the moduli space is a scheme/ algebraic space/ stack; the properties we investigate in the next sections are about the moduli functor, so it doesn't really matter if there is a geometric object representing it. However, in the second half of the notes we will actually show that there is scheme representing the moduli functor described above, where we further identify S-equivalent semistable sheaves.

Next, we recall some result about boundedness that will be used in both sections 2 and 4. First, recall that a family of isomorphism classes of coherent sheaves on X is called bounded if there exists a scheme S of finite type over k and a coherent $\mathcal{O}_{S\times X}$ sheaf $\mathcal{F}$ such that the given family is contained in the set $\{\mathcal{F}_s | s \in S\}$. A very useful characterization of boundedness can be done in terms of the Mumford Castlenuovo regularity, which is defined for a coherent sheaf F as

$$\rho(F) := \inf\{m | H^i(X, F(m-i)) = 0, \text{for all } i \geq 0\}.$$

A criterion due to Mumford says that a family of sheaves $\{F_i\}$ is bounded if and only if the set of Hilbert polynomials $\{P(F_i)\}$ is finite and there is a uniform bound for the Mumford-Castelnuovo regularity of all the sheaves F_i in the family.

We will use inductive arguments to establish boundedness, and for that we will want a good notion of transversality: if E is "nice", we want $E|_H$ its restriction to a hyperplane section to be "nice" as well. The correct notion of transversality in our case is that of $F-$regularity. Recall that a hyperplane $s \in H^0(X, O(1))$ is called $F-$regular if $F \otimes \mathcal{O}(-1) \to F$ given by multiplication by s is injective. The hyperplane determined by s is $F-$regular if it does not contain any of the (finitely many) associated points of F. We say that a sequence of hyperplanes $s_1, ..., s_d \in H^0(X, \mathcal{O}(1))$ is $F-$regular if s_i is $F/(s_1, ..., s_{i-1})(F \otimes \mathcal{O}(-1))-$ regular for all $1 \leq i \leq d$. Also, we introduce some notation which will be used throughout the notes: given a sheaf F and hyperplanes $H_1, ..., H_d$, we denote by F_i the restriction of F to the intersection $H_1 \cap H_2 \cap ... \cap H_{d-i}$. This condition means that H_i does not contain any of the associated points of the restriction of F to the intersection $H_1 \cap ... \cap H_{i-1}$. Next, we state Kleiman's criterion, which will be used in inductive arguments related to boundedness.

Theorem 2.1. *[3, Theorem 1.7.8] Let $\{F_i\}$ be a family of coherent sheaves on X with the same Hilbert polynomial P. Then this family is bounded if and only if there exist constants C_j, for $0 \leq j \leq d$, such that for every element F of the family, there exists a $F-$regular sequence of hyperplane sections $H_1, ..., H_d$ such that $h^0(F_i) \leq C_i$, for all $0 \leq i \leq d$.*

As we were saying, Kleiman's criterion will be used in inductive arguments. Thus, it would be good to have a general result for semistable sheaves on a curve. This is given by the following:

Lemma 2.2. The family of semistable sheaves with fixed Hilbert polynomial on a smooth projective curve is bounded.

Proof. The family of zero-dimensional sheaves is certainly bounded, so we focus on one-dimensional sheaves. We want to bound the Mumford-Castelnuovo regularity, so we want to find an m for which

$$H^1(X, F(m-1)) = \mathrm{Hom}(F, \omega_X(1-m))^\vee = 0$$

in terms of the Hilbert polynomial only.

But both F and $\omega_X(1-m)$ are semistable, and we know that there are no maps from a semistable sheaf to another if the first one has larger Hilbert polynomial. It is clear that we can choose m such that this happens. □

Next, we discuss a theorem of Grothendieck which will be used in establishing openness.

Theorem 2.3. *[3, (proof of) Theorem 1.7.9] Let X be projective scheme, $\mathcal{O}(1)$ ample line bundle, and E a $d-$dimensional sheaf with Hilbert polynomial P and Castelnuovo-Mumford regularity ρ, and let $\mu_0 > 0$ be a real number. The family of purely $d-$dimensional quotients $E \twoheadrightarrow F$ with $\mu'(F) \leq \mu_0$ is bounded and the regularity of E is bounded by ρ, P, and μ_0 only.*

3 Openness

Suppose we are given a flat family $\{F_s\}$ of $d-$dimensional sheaves with Hilbert polynomial P on the fibers of a projective morphism $f : X \to S$ and that $\mathcal{O}(1)$ is an $f-$ample invertible sheaf. In this section, we show that the locus $s \in S$ for which F_s is semistable is open. This will establish:

Theorem 3.1. *Semistability is open in flat families.*

Proof. Say that F_s is a member of the flat family and that it is not semistable. This means that there exists a proper purely d dimensional quotient $F_s \twoheadrightarrow E$ such that $p(E) \leq p(F_s)$. This means, in particular, that $\mu'(E) \leq \mu'(F_s)$. We are looking at quotients of F_s whose μ' is bounded, so we can invoke Grothedieck's theorem 2.3 to deduce that this family is bounded, and that the regularity $\rho(E)$ is bounded in function of $\rho(F_s)$, and the Hilbert polynomial $P(F_s)$, which is the same for all of them. The regularity of F_s is bounded for

$s \in S$ because F is a bounded family. This means that $\rho(E)$ for all quotients E of F_s, where s varies in S, is bounded, so the family of such quotients is bounded. Now, using the Mumford-Castelnuovo criterion we deduce that there are only finitely many Hilbert polynomials in the set $\{P(E)\}$, where E is a quotient of a fiber F_s as above. Now, every such quotient will imply that there exists a polynomial $g < p$ such that the point $s \in S$ is in the image of $\pi : \mathrm{Quot}_{X/S}(F, G) \to S$, for example, by base-changing via the map $s \to S$. Since π is proper, the image is a closed subset of S. Also, there are finitely many possibilities for the Hilbert polynomial G, and F_s is semistable precisely when s is in the complement of the finite union of these closed sets, which is open. □

4 Boundedness

Now, we want to prove boundedness for the family of semistable sheaves. Boundedness will be used later in realizing semistable sheaves as points of a Quot scheme. It is also the point where it matters that we work in characteristic 0, as the proof in characteristic p is more involved and uses another notion of stability (all pullbacks via Frobenius should be Gieseker semistable).

It is natural to proceed by induction, given Kleiman's criterion. Given a regular sequence of hyperplane sections $H_1, ..., H_d$, define $X_v = H_1 \cap ... \cap H_{d-v}$, and let F_v be the restriction of F to X_v. To prove boundedness, it is enough to bound $h^0(X_v, F_v)$ in function of d, the dimension of the support of F, and P, the Hilbert polynomial of F. It would be nice if the restriction of F to a general hyperplane section was semistable, then we would have been able to use induction right away. Unfortunately, this is not true, but one has control over how bad $F|_H$ fails to be semistable if one uses $\mu-$stability. This is the content of the Grauert-Mulich theorem. But, before we state it, we need a definition. For F a non necessarily $\mu-$stable sheaf, arrange the slopes of the Harder-Narasimhan filtration in increasing order $\mu_1 \geq ... \geq \mu_s$. So, we have d hyperplane sections and s Harder- Narasimhan factors (we would like $s = 1$, i.e. the restriction F_v to X_v is semistable, but, as we said above, this might not happen). Define

$$\delta(F) := \max \{\mu_i - \mu_{i+1} | i = 1, s-1\},$$

a quantity which measures how far F is from being (semi)stable.

Theorem 4.1. *[3, Theorem 3.1.2] Let X be a normal projective variety with very ample sheaf $\mathcal{O}(1)$. Let F be a $\mu-$semistable sheaf and let $H_1, ..., H_d$*

be some hyperplane sections. Define F_0 *to be the restriction of* F *to their intersection. Then for a generic choice of hyperplane sections,*

$$0 \leq \delta(F_0) \leq \deg(X),$$

where $\deg(X)$ *is, as usual, the self-intersection number of* $\mathcal{O}(1)$ *taken* $\dim(X)$ *times.*

Let's see an example of use of this theorem (this is actually the original theorem Grauert and Mulich proved):

Theorem 4.2. *[3, Theorem 3.0.1] Let* E *be a* $\mu-$*semistable locally free sheaf of rank* r *on complex projective space* $\mathbb{P}^n$*. If* L *is a general line in* $\mathbb{P}^n$ *and* $E|_L \cong \mathcal{O}_L(b_1) \oplus \mathcal{O}_L(b_2) \oplus ... \oplus \mathcal{O}_L(b_r)$*, with integers* $b_1 \geq ... \geq b_r$*, then*

$$0 \leq b_i - b_{i+1} \leq 1$$

for all $1 \leq i \leq r-1$*.*

One can actually formulate a more general statement involving arbitrary degree hypersurfaces instead of hyperplane sections. There are also other (stronger) theorems with the same flavour that can be used as replacements for Grauert- Mulich. For example, Flanner has proved that the restriction of a $\mu-$semistable sheaf F to a general degree d hypersurface is $\mu-$semistable in characteristic zero, for d explicitly computable in function of invariants of F. Mehta and Ramanathan proved the same statement over arbitrary characteristic, but with no control over the degree of the hyerpsurface, so their result did not imply boundedness of the moduli space of semistable sheaves in characteristic p. For more theorems about restrictions to hypersurfaces see [3, Chapter 7] and [4].

The other important ingredient is the Le Potier- Simpson theorem, which gives a bound for $h^0(X_v, F_v)$ in function of various invariants of F. We will also need this estimate when we characterize semistable sheaves in Section 7.

Theorem 4.3. *[3, Theorem 3.3.1] Let* X *be a projective variety,* $\mathcal{O}(1)$ *an ample line bundle,* F *a* $d-$*dimensional pure sheaf of multiplicity* r*. For a sequence of hyperplanes* H_i*,* $1 \leq i \leq d$*, define* $X_v = H_1 \cap ... \cap H_{d-v}$ *and* F_v *the restriction of* F *to* X_v*, for all* $1 \leq v \leq d$*. Then, there exists an* $F-$*regular sequence of hyperplanes* H_i*,* $1 \leq i \leq d$*, such that*

$$h^0(X_v, F_v) \leq \frac{r}{v!}\left[\mu'_m(F) + \frac{r(r+d)}{2} - 1\right]_+^v,$$

where $[x]_+ := \max\{x, 0\}$ *and where* $\mu'_m(F)$ *is the maximal slope that appears in the Harder- Narasimhan filtration of* F*.*

As we were saying above, it only matters that we can find a sequence of hyperplanes such that $h^0(X_v, F_v)$ is bounded in function of invariants of F coming from the Hilbert polynomial, the exact bound given by the theorem does not really matter for us.

Proof. We prove it only in the torsion free case. The idea is the following: we first bound $h^0(X_v, F_v)$ in function of F_1. It should not come as a surprise that this can be done: we lose control more and more on the Harder-Narasimhan factors as we take more hyperplane sections, but we know that the degree remains constant, so $\mu'_m(F_1)$ should dominate all the terms. The second part involves bounding $\mu'_m(F_1)$ in function of $\mu'_m(F)$, which should seem surprising at first, considering what we have just said, but not after seeing the Grauert-Mulich theorem, which controls the slope of the Harder-Narasimhan factors on F_1. We thus split the proof in two cases:

Step 1. We show by induction on v that

$$h^0(X_v, F_v) \leq \frac{\mathrm{rk}D}{v!}\left[\frac{\mu_m(F_1)}{D} + v\right]_+^v,$$

where D is the degree of X and rk is the rank of F, $r = \mathrm{rk}D$. For $v = 1$, we have

$$h^0(X_1, F_1) \leq \sum_i h^0(X_1, gr_i^{NH}(F_1)),$$

and we can assume that $\mu_m(F_1) = \mu(F_1)$, i.e. that F_1 is semistable. Further, we know by boundedness for semistable sheaves on a curve that $h^0(X_1, F_1(-l)) = 0$ for $l > \frac{\mu(F_1)}{D}$. We also have the estimate

$$h^0(X_1, F_1) \leq h^0(X_1, F_1(-l)) + \mathrm{rk}lD.$$

Thus, for $l = \lfloor \frac{\mu(F_1)}{D} + 1 \rfloor$ we get the bound claimed above.

For the inductive step, use the exact sequences

$$0 \to F_v(-k-1) \to F_v(-k) \to F_{v-1}(-k) \to 0$$

to obtain

$$h^0(X_v, F_v) \leq \sum_i h^0(X_{v-1}, F_{v-1}(-i)).$$

Now, the result follows from the induction hypothesis and some elementary computations.

Step 2. It is enough to show that

$$\mu_m(F_1) + vD \leq \mu_m(F) + vD + \frac{(\mathrm{rk}-1)D}{2}.$$

For this, we can assume that F is semistable, otherwise choose the quotient which gives $\mu_m(F)$ and look at its restriction to X_1. Assume that the slopes and the ranks of the HN factors for F_1 are $\mu_1 \geq \mu_2 \geq ... \geq \mu_s$ and $\mathrm{rk}_1, ..., \mathrm{rk}_s$. The Grauert-Mulich theorem says that $\mu_i - \mu_{i+1} \leq D$, for all $1 \leq i \leq s$, and thus

$$\mu(F) = \sum_{i=1}^{s} \frac{\mathrm{rk}_i}{\mathrm{rk}} \mu_i \geq \mu_1 - \sum_{i=1}^{s} (i-1) \frac{\mathrm{rk}_i}{\mathrm{rk}} D,$$

which can be bounded below by

$$\mu_1 - \frac{D}{\mathrm{rk}} \sum_{i=1}^{\mathrm{rk}} (i-1) = \mu_m(F_1) - \frac{D(\mathrm{rk}-1)}{2}.$$

This ends the proof in the torsion free case. □

Now, we are ready to prove boundedness for semistable sheaves.

Theorem 4.4. *Let $f : X \to S$ be a projective morphism of schemes of finite type and let $\mathcal{O}(1)$ be an $f-$ample line bundle. Let P be a polynomial of degree d, and let μ_0 be a rational number. Then the family of purely $d-$dimensional sheaves on the fibers of f with Hilbert polynomial P and maximal slope $\mu'_{\max} \leq \mu_0$ is bounded. In particular, the family of semistable sheaves on the fibres of f with Hilbert polynomial P is bounded.*

Proof. We reduce to the case $S = \mathrm{Spec}(k)$ and $X = \mathbb{P}^n$. The Le Potier-Simpson estimate says that for every purely $d-$dimensional coherent sheaf F we can find a sequence of $F-$ regular hyperplanes such that $h^0(F_v) \leq C$, for $0 \leq i \leq d$, where C is a constant depending only on the dimension and the degree of X and the multiplicity and slope of F. For a semistable sheaf, these depend on the Hilbert polynomial only. Now, boundedness for semistable sheaves follows from Kleiman's criterion. □

5 Properness

Recall the valuative criterion for properness:

Theorem 5.1. *[2, Theorem II.4.7 and Exercise II.4.11]*

Let $f : X \to Y$ be a finite type morphism of noetherian schemes. Then f is proper if and only if for every discrete valuation ring R with maximal ideal $(\pi), \pi \in R$ and quotient field K, and for every morphism of Spec(K) to X and for every morphism Spec(R) to Y, there exists a unique morphism Spec$(R) \to X$ making the following diagram commutative:

$$\begin{array}{ccc} Spec\,(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Spec\,(R) & \longrightarrow & Y \end{array}$$

Denote by $k = R/(\pi)$ the residue field of R. We do not expect separatedness in general for the moduli functor of semistable sheaves. This can be seen using the example used to show there is no fine moduli space representing the moduli functor. Let's recall it: if there exist semistable sheaves F_1 and F_2 with $\mathrm{Ext}^1(F_2, F_1) \neq 0$, choose F a non-trivial extension. We can construct a flat family $\mathcal{F}$ of semistable sheaves on $\mathbb{A}^1$ such that $\mathcal{F}_0 \cong F_1 \oplus F_2$ and $\mathcal{F}_t \cong F$ by taking the line in $\mathrm{Ext}^1(F_2, F_1)$ corresponding to F. We also have the constant $F_1 \oplus F_2$ family on $\mathbb{A}^1$. Both of these families are isomorphic on $\mathbb{A}^1 - 0$, but have different fibers over 0, which means there are at least two diagonal maps in the above diagram which make it commutative.

This means we cannot expect separatedness if we do not identify S-equivalence classes for semistable sheaves. However, we expect separatedness for stable sheaves and an extension property (filling the diagonal map in the valuative criterion diagram) for semistable sheaves. Both of these results will follow as consequences of the semi-continuity theorem and of the following theorem (extension of a result of Langton):

Theorem 5.2. *[3, Theorem 2.B.1] Let F be an $R-$flat family of $d-$dimensional coherent sheaves on X such that $F_K = F \otimes K$ is a semistable sheaf. Then there exists a subsheaf $E \subset F$ such that $E_K = F_K$ and E_k is a semistable sheaf.*

Because a subsheaf of a flat sheaf is flat, E is flat over $\mathrm{Spec}(R)$. This proposition implies that the moduli of stable sheaves is separated. Indeed, using once again the valuative criterion, we have to show that F_K has exactly one extension over R. Suppose F and F' are two different extensions. Then, by the semi-continuity property [1, Satz 3(i)] for Hom of sheaves, we get a non-zero map $F_k \to F'_k$. But they are both stable sheaves with the same Hilbert polynomial (that of F_K), so this is not possible.

Proof. The rough idea of the proof is as follows: we construct E one step at the time, working in the categories $\mathrm{Coh}_{d,e}$. In case there is a value e such that we cannot extend it further, we will get a destabilizing sheaf G of F_k. We will try to modify the family F over $\mathrm{Spec}(k)$ so that the new family is semistable. Assuming this cannot be done, we construct infinite chains of maximal destabilizing sheaves of F_k. We will use these chains of sheaves to construct flat quotients of $F \otimes R/\pi^n R$ with Hilbert polynomial $p(G) < p(F)$, for every $n \geq 1$. This will imply that there is actually such a destabilizing flat quotient over R, and thus that F_K admits a destabilizing subsheaf, which will contradict the hypothesis that F_K is semistable over K.

As advertised above, we will use induction in the following way: if F is as above and F_k is semistable in $\mathrm{Coh}_{d,e+1}$, then there exists a sheaf $E \subset F$ such that $E_K = F_K$ and E_k is semistable in $\mathrm{Coh}_{d,e}$. The theorem follows by descending induction on e, and the base case $e = d$ is empty. So, fix some $e \leq d-1$ and suppose the claim was false for e. Define a descending sequence of sheaves $F = F^0 \supset F^1 \supset ...$ with $F^n_K = F_K$ and F^n_k not semistable in $\mathrm{Coh}_{d,e}$ as follows. Suppose F^n has been defined, then let B^n be the maximal destabilizing subsheaf of F^n_k. Define further $G^n = F^n_k/B^n$ and let F^{n+1} be the kernel of the composite homomorphism $F^n \to F^n_k \to G^n$. As a submodule of an $R-$flat sheaf, F^{n+1} is $R-$flat again. Then

$$0 \to B^n \to F^n_k \to G^n \to 0 \tag{5.1}$$

is exact by definition. Further,

$$0 \to F^{n+1} \to F^n \to G^n \to 0$$

is exact, so by restricting over $\mathrm{Spec}(k)$, we get that

$$\mathrm{Tor}^R_1(F^n, k) \to \mathrm{Tor}^R_1(G^n, k) \to F^{n+1}_k \to F^n_k \to G^n \to 0.$$

Now, $\mathrm{Tor}^R_1(F^n, k) = 0$ because F^n is flat. Further, using the exact sequence

$$0 \to R \xrightarrow{\pi} R \to k \to 0$$

we compute that $\mathrm{Tor}^R_1(G^n, k) = G^n$. Thus, we deduce that

$$0 \to G^n \to F^{n+1}_k \to B^n \to 0. \tag{5.2}$$

Observe that both G^n and B^{n+1} are subsheaves of F^{n+1}_k. Now, the plan is to show that $G^n \cap B^{n+1} = 0$ for big enough n, which will imply that $B^{n+1} \subset B^n$ and $G^n \subset G^{n+1}$. We will see that this implies that the sequences 5.1 and 5.2 split. Define $C^n = B^{n+1} \cap G^n$; then C^n is a subsheaf of B^n. We will use the notation $p_{max}(F)$ to denote the Hilbert polynomial of the maximal destabilizing subsheaf of a given sheaf F. Observe that

$$p(C^n) \leq p_{max}(G^n) < p(F^n_k) \leq p(B^{n+1}) \bmod \mathbb{Q}[T]_{e-1},$$

where the first inequality is true by the definition of p_{max} and because $C^n \subset G^n$. The second inequality is true by the choice of B^n as the maximal destabilizer sheaf of F^n_k: if there would have been a subsheaf $H \subset G^n$

such that $p(H) > p(F_k^n)$, then looking at the preimage of H in F_k^n we find a subsheaf H' containing B^n which has $p(H') > p(F_k^n)$, contradicting the maximality of B^n. The third one follows in a similar way from the definition of B^{n+1}. Since B^{n+1}/C^n is isomorphic to a nonzero submodule of B^n it follows that

$$p_{d,e}(B^{n+1}) \leq p_{d,e}(B^{n+1}/C^n) \leq p_{d,e}(B^n) \tag{5.3}$$

with equality if and only if $C^n = 0$. We know that $p_{d,e}(B^n) > p_{d,e}(F_k^n)$. However, recall that F_k is semistable in $\mathrm{Coh}_{d,e+1}$, so we must have $p_{d,e+1}(B^n) = p_{d,e+1}(F_k) = p_{d,e+1}(G^n)$ for all n, which means that

$$p_{d,e}(B^n) - p_{d,e}(F_k) = \beta_n T^e \bmod \mathbb{Q}[T]_{e-1}$$

for a rational number β_n. Since $p_{d,e}(B^n) > p_{d,e}(F_k^n)$ it follows that $\beta_n > 0$. The sequence β_n is decreasing, bounded below, and it is contained in the lattice $\frac{1}{r!}\mathbb{Z} \subset \mathbb{Q}$, so it has to become stationary; we can actually assume it is constant from the beginning. This implies by the equality case of inequality 5.3 that $C^n = 0$. In particular, we have $B^{n+1} \subset B^n$ and $G^n \subset G^{n+1}$. Now, this implies that $P(B^0) \equiv p(B^1) \equiv ... \bmod \mathbb{Q}[T]_{e-1}$. From the exact sequence 5.1,

$$P(G^n) = P(F_k^n) - P(B^n),$$

and also $P(F_k^n) = P(F_K^n) = P(F_K)$, because F is flat over $\mathrm{Spec}(R)$. It follows that $P(G^0) \equiv P(G^1) \equiv ... \bmod \mathbb{Q}[T]_{e-1}$, and thus that $G_0 \subset G_1 \subset ...$ is a sequence of purely $d-$dimensional sheaves which are isomorphic in dimension $\geq d-1$. Now, two subsheaves with the same support of dimension d isomorphic in dimensions $\geq d-1$ have the same reflexive hull (result which is a corollary of [3, Section 1.1]). This implies that the sheaves G^n have the same reflexive hull, and thus we can regard them as being subsheaves of a fixed sheaf (this common reflexive hull). The inclusions become eventually isomorphisms, and we assume once again that happens for $n = 0$.

The map $G^n \to F_k^{n+1} \to G^{n+1}$ obtained from combining the maps from the exact sequences 5.1 and 5.2 is thus an isomorphism, thus the short exact sequences 5.1 and 5.2 split. We will use the short notations $B = B^n$, $G = G^n$ from now on. We have that $F_k^n = B \oplus G$. Define $Q^n = F/F^n$, $n \geq 0$. Now, we want to show that Q^n is an $R/\pi^n R$ flat quotient of $F/\pi^n F$ with Hilbert polynomial $P(G)$. The first step in doing this is showing that $Q_k^n \cong G$. To see this, observe that

$$F_k^{n+1} \to F_k^n \to Q_k^n \to 0. \tag{5.4}$$

The map $F_k^{n+1} \to F_k$ factors through the maps $F_k^n = B \oplus G \to F_k^{n-1} = B \oplus G$. Now, using the definiton of F^n as the kernel of $F^{n-1} \to G^{n-1} \to 0$, we find that

$$F_k^n = B \oplus G \to F_k^{n-1} = B \oplus G \to G \to 0,$$

which shows that the map $F_k^n \to F_k^{n-1}$ is actually $B \oplus G \xrightarrow{\mathrm{id} \oplus 0} B \oplus G$. Coming back in the sequence 5.4, we find that $Q_k^n \cong G$. Using this result and the exact sequence

$$0 \to G \to Q^{n+1} \to Q^n \to 0,$$

we can deduce that Q^n is actually an $R/\pi^n R$ flat module. It is also a quotient of $F/\pi^n F$, by construction. All in all, this implies that the image of the proper map $\pi : \mathrm{Quot}_{X_R/R}(F, P(G)) \to \mathrm{Spec}(R)$ contains the closed subscheme $\mathrm{Spec}(R/\pi^n R)$. Thus, the proper map

$$\pi : \mathrm{Quot}_{X_R/R}(F, P(G)) \to \mathrm{Spec}(R)$$

has to be surjective. By base change, $\mathrm{Quot}(F_K, P(G)) \to \mathrm{Spec}\,(K)$ is surjective, which implies that F_K also admits a destabilizing quotient with Hilbert polynomial $p(G) < p(F)$. This contradicts the assumption that F_K is semistable, and ends the proof.

□

6 Setting up the GIT construction

As usual, X is a projective scheme with an ample line bundle $\mathcal{O}(1)$. Fix a polynomial $p \in \mathbb{Q}[X]$. Recall the definition of the moduli functor from section 2: $\Phi : \mathrm{Sch/k}^{op} \to \mathrm{Sets}$, $\Phi(S)$ is the set of isomorphism classes of $S-$flat families of semistable sheaves on X with Hilbert polynomial P up to equivalence, where we say that two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there exists a line bundle on S such that $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$.

We explain how we can regard the semistable sheaves as points of a certain Quot scheme, invariant under the action of $SL(V)$ for a k vector space V to be defined in a few lines. First, we know that the family of semistable sheaves on X with Hilbert polynomial equal to P is bounded. This means that there exists an integer m (depending on P only) such that every such F is $m-$regular. Hence, $F(m)$ is globally generated and $h^0(F(m)) = P(m)$. Let V be a $k-$vector space of dimension $P(m)$; one can think of V as $H^0(F \otimes \mathcal{O}(m))$ for F a semistable sheaf. There exists a surjection $V \otimes O_X(-m) \to F$, and thus a point of $\mathrm{Quot}(V \otimes O_X(-m), P)$. We will use the shorther notation Quot for the scheme $\mathrm{Quot}(V \otimes O_X(-m), P)$ in the rest of these notes.

This point is contained in the open subset $R \subset \mathrm{Quot}$ of all the quotients $V \otimes O_X(-m) \to F$ where the induced map $V \to H^0(F(m))$ is an isomorphism. Its closure $\bar{R}$ will play an important role in our arguments. Denote by $R^s \subset R$ the open subset of stable subsheaves. All semistable sheaves with Hilbert polynomial P appear as points of Quot, but with an ambiguity arising from the choice of basis of $H^0(F(m))$. The group $GL(V)$ acts by composition on Quot. The open subset R is invariant under this action and isomorphism classes of semistable sheaves are given by the set $R(k)/GL(V)(k)$.

Before discussing further the construction of the moduli of semistable sheaves, let's recall how GIT can be used to construct quotients of a projective scheme by a reductive algebraic group as projective schemes. For a projective scheme X with an action of a reductive group G and L a $G-$linearized ample line bundle, one defines certain $G-$invariant open subsets of X, possibly empty, of stable and semistable points $X^s \subset X^{ss}$. For the definition of (semi)stable points in the context of GIT and the definitions of categorical, good, and geometric quotient, see the previous set of notes or $[3, Chapter 4.2]$.

Theorem 6.1. *[3, Theorem 4.2.10] Let G be a reductive group acting on a projective scheme X with a $G-$linearized ample line bundle L. Then there exists a projective scheme Y and a morphism $\pi : X^{ss}(L) \to Y$ such that π is a universal good quotient for the $G-$action. Moreover, there is an open subset $Y^s \subset Y$ such that for $X^s(L) = \pi^{-1}(Y^s)$, the map $\pi : X^s(L) \to Y^s$ is a universal geometric quotient.*

We will eventually want to use the above theorem for the projective scheme $X = \bar{R}$ and the reductive group $G = SL(V)$. However, before we can apply the above theorem, we first have to find a $G-$linearized ample line bundle on $\bar{R}$. It is actually enough to find one over Quot.

One can show that the center $Z \subset GL(V)$ is contained in the stabilizer of any point in Quot. Instead of actions of $GL(V)$ we will consider actions of $PGL(V)$ and $SL(V)$. It is actually a little easier to find one for $SL(V)$, so we will work with this group. Now, recall that we have constructed the Quot scheme as a subscheme of a certain Grassmannian [3, Section 2.2]. Indeed, for a projective morphism $f : X \to S$, for a general coherent $\mathcal{O}_X-$module $\mathcal{H}$ and for a Hilbert polynomial P, we showed that for large l, there is a closed immersion

$$\mathrm{Quot}_{X/S}(\mathcal{H}, P) \to \mathrm{Grass}_S(f_*\mathcal{H}(l), P(l)).$$

Recall the standard proof that the Grassmannian is projective using the Plucker embedding (for more details, see [3, Section 2.2]). We can pull back the tautological line bundle from the projective space to get a very ample line bundle on the Grassmannian, and thus on the Quot scheme. All in all, this line bundle on Quot, in our particular case, is

$$L_l := \det(p_*(\mathcal{U} \otimes q^*\mathcal{O}_X(l))),$$

where $p : \mathrm{Quot} \times X \to \mathrm{Quot}$ and $q : \mathrm{Quot} \times X \to X$ are the projections onto the two factors, and where $\mathcal{U}$ is the universal quotient sheaf on $\mathrm{Quot} \times X$. One can show that L_l has a natural $GL(V)-$linearization (which by restriction will induce a natural $SL(V)-$linearization) using the results discussed in Lecture 3.

Now we have all the ingredients required by Theorem 6.1 to construct a quotient: we take X to be $\bar{R}$ the closure of the points where the induced map $V \to H^0(F(m))$ is an isomorphism, G to be $SL(V)$, and L to be L_l defined above. Then Theorem 6.1 produces for us a new projective scheme, and there is a priori no reason why this should be related to a potential moduli scheme of semistable sheaves. The main theorem of this lecture is that ***(semi)stable points for the above GIT setup correspond to (semi)stable sheaves.*** This means that there exist a good quotient of the action of $SL(V)$ on R. Further, it also gives a correspondence between the closed points of the quotient and S-equivalence classes of semistable sheaves. This means that the projective scheme produced by Theorem 6.1 is precisely the scheme we were looking after, the moduli scheme of semistable sheaves! The exact form of the result we are going to discuss in the next section is the following:

Theorem 6.2. *Let $l \gg m \gg 0$ sufficiently large integers. Then $R = \overline{R}^{ss}(L_l)$ and $R^s = \overline{R}^{s}(L_l)$. Moreover, the closure of the orbits of two points $V \otimes O_X(-m) \to F_1$ and $V \otimes O_X(-m) \to F_2$ intersect if and only if $gr^{JH}(F_1) \cong gr^{JH}(F_2)$. The orbit of a point $V \otimes O_X(-m) \to F$ is closed if and only if it is polystable.*

Recall that $gr^{JH}(F)$ is the direct sum of the quotients appearing in the Jordan-Holder filtration of a sheaf $F-$ for more details, check the notes for Lecture 1. One can prove [3, Lemma 4.3.1] that a categorical quotient of R by $SL(V)$ corepresents the moduli functor. Thus, as explained in the above paragraph, this proves:

Theorem 6.3. *There is a projective scheme $M(\mathcal{O}_X(1), P)$ that universally corepresents the moduli functor Φ. Closed points in $M(\mathcal{O}_X(1), P)$ are in bijection with S-equivalence classes of semistable sheaves with Hilbert polynomial P. Moreover, there is an open subset $M^s(\mathcal{O}_X(1), P)$ that universally corepresents the moduli functor Φ^s.*

7 Proof of theorem 6.2

Theorem 6.2 says that (semi)stability for sheaves is equivalent to (semi)stability in the GIT sense (for m and l large enough). This means we should characterize GIT semistable points and semistable sheaves and try to show that these characterizations are the same. There is a good way to test whether a point is semistable or not, namely the Hilbert-Mumford criterion. So, the plan is the following: we test test whether a point $V \otimes O_X(-m) \to F$ is GIT (semi)stable using the Hilbert-Mumford criterion, and we discover that this characterizes GIT semistability in function of some inequalities involving the number of global sections of subsheaves $F' \subset F$. After that, we will try to find a similar characterizations of semistable sheaves, and we will discuss Le Potier's theorem, which does exactly that. Finally, we will explain how these ingredients can be put together to prove Theorem 6.2.

So, let's start with $\rho : V \otimes O_X(-m) \to F$, a closed point in $\overline{R}$, where recall that $R \subset \text{Quot}$ is the open subset of the Quot scheme parametrizing points where $V \to H^0(F(m))$ is an isomorphism. As we were saying in the above paragraph, we test whether this point is semistable using the Hilbert-Mumford criterion. Let's briefly recall what this criterion says in the general GIT context, where X is a projective scheme and G is a reductive group acting on X. For a more throughout analysis, see [3, Section 3.2]. Given a one parameter subgroup $\lambda : \mathbb{G}_m \to G$, we get an action of $\mathbb{G}_m$ on X. Since X is projective, the orbit map $\mathbb{G}_m \to X$, $t \to \lambda(t)x$ extends in a unique way to a morphism $f : \mathbb{A}^1 \to X$ with $f(0)$ fixed point of the action on $\mathbb{G}_m$ on X via λ. This means that $\mathbb{G}_m$ acts on the fiber $L(f(0))$ over $f(0)$ with a certain weight r, and define $\mu(x, \lambda) := -r$.

Lemma 7.1 (Hilbert-Mumford)**.** A point $x \in X$ is (semi)stable if and only if for any non-trivial one parameter subgroup $\lambda : \mathbb{G}_m \to G$,

$$\mu(x, \lambda)(\geq) > 0.$$

In order to apply the Hilbert- Mumford criterion we need to determine the limit point $\lim_{t\to 0}[\rho]\lambda(t)$ for the action of any one parameter subgroup $\lambda : \mathbb{G}_m \to SL(V)$ on $[\rho] = [\rho : V \otimes O_X(-m) \to F]$ a point in $\overline{R}$; λ is completely determined by the decomposition $V = \oplus_{n\in Z} V_n$ into weight pieces V_n of weight n. Define ascending filtrations of V and F by $V_{\leq n} = \oplus_{s\leq n} V_s$ and by $F_{\leq n} = \rho(V_{\leq n} \otimes \mathcal{O}(-m)) \subset F$. Then ρ induces surjections $\rho_n : V_n \otimes O(-m) \to F_n = F_{\leq n}/F_{\leq n-1}$. Summing over all weights we get a closed point

$$\overline{\rho} := \oplus \rho_n : V \otimes \mathcal{O}(-m) \to \overline{F} := \oplus F_n$$

in the Quot scheme in question. One can show the following:

Lemma 7.2. The limit $\lim_{t\to 0}[\rho]\lambda(t)$ is $[\overline{\rho}]$.

Now, we can compute the weight of the action of $\mathbb{G}_m$ via the character λ on the fiber of L_l at the point $[\overline{\rho}]$. The Quot scheme is bounded, so we can choose l such that all $H^i(F(l)) = 0$, for all elements F of Quot and for $i \geq 1$. In particular, $P(F, l) = H^0(F(l))$. Indeed, $\overline{F} = \oplus F_n$ decomposes in subsheaves on which $\mathbb{G}_m$ acts via the character $t \to t^n$, hence $\mathbb{G}_m$ will act by weight n on $H^0(F_n(l))$. In particular, it acts on the determinant of the complex with cohomology groups $H^i(F_n(l))$ with weight $nP(F_n, l)$. Looking at the definition of L_l, we see that

$$L_l(\overline{\rho}) = \otimes_n \det(H^0(F_n(l))),$$

which means that λ acts on $L_l(\overline{\rho})$ with weight $\sum_n nP(F_n, l)$. Recall that the Hilbert-Mumford criterion says, in this particular case, that

$$\sum_n nP(F_n, l) \leq 0$$

for all one parameter subgroups $\lambda : \mathbb{G}_m \to G$.

Now, we can rewrite this weight using the fact that the determinant of λ is 1, which implies $\sum n \dim(V_n) = 0$. After some easy manipulations, the weight becomes

$$\sum_n nP(F_n, l) = -\frac{1}{\dim(V)} \sum_{n \in \mathbb{Z}} (\dim(V) P(F_{\leq n}, l) - \dim(V_{\leq n}) P(F, l)).$$

This implies:

Lemma 7.3. A closed point $\rho : V \otimes O_X(-m) \to F$ in $\overline{R}$ is (semi)stable if and only if for all non-trivial proper linear subspaces $V' \subset V$ and the induced subsheaf $F' \subset F$ generated by V' we have:

$$\dim(V) P(F', l) (\geq) > \dim(V') P(F, l).$$

One can actually prove a variant of the lemma where the inequality to be checked is in function of the Hilbert polynomial only, and this is what will be used in the proof of Theorem 6.2.

Lemma 7.4. If l is sufficiently large, a closed point $\rho : V \otimes O_X(-m) \to F$ in $\overline{R}$ is (semi)stable if and only if for all coherent subsheaves $F' \subset F$ and $V' = V \cap H^0(F'(m))$, the following inequality holds:

$$\dim(V) P(F') (\geq) > \dim(V') P(F).$$

Proof. First, we remark that for l large enough, the inequality stated above is equivalent to the estimate

$$\dim(V) P(F', l) (\geq) > \dim(V') P(F, l),$$

which is similar to the form of Lemma 7.3. The family of subsheaves F' generated by V' is bounded, so there are finitely many possible Hilbert polynomials $P(F')$, which means that for large l the conditions of the two lemmas are equivalent. Moreover, if F' is generated by V', then $V' \subset V \cap H^0(F'(m))$, and conversely, if F' is an arbitrary subsheaf of F and $V' = V \cap H^0(F'(m))$, then the subsheaf of F generated by V' is contained in F'. □

Now, we have a good description of what are the GIT (semi)stable points of the Quot scheme. Let's see what would it mean for a semistable sheaf to be GIT semistable sheaf− we will a imprecise and vague in the following argument. We would like to have $\dim(V)P(F') \geq \dim(V')P(F)$, for any subsheaf $F' \subset F$ with multiplicity $0 < r' < r$, where $V' = H^0(F'(m))$. Thus, we would like to prove

$$h^0(F(m))r'p(F') \geq h^0(F'(m))rp(F).$$

However, for semistable sheaves we know that $p(F') \leq p(F)$, so this looks like it is going in the other direction. However, they are both monic polynomials of the same degree, so maybe we can prove

$$h^0(F(m))r' \geq h^0(F'(m))r$$

and show that equality occurs precisley when $p(F') = p(F)$.

Now, if we choose m large enough, $h^0(F(m)) = rp(F, m)$ and $h^0(F'(m)) = r'p(F', m)$, and thus the inequality becomes

$$rr'p(F, m) \geq rr'p(F', m),$$

which is true for m big enough depending on F and F'. However, we have to choose m depending on the Hilbert polynomial P alone (which we can do, because the family of semistable sheaves with Hilbert polynomial P is bounded), and not on all subsheaves $F' \subset F$ of such sheaves F. One can easily see that this family is not bounded. This means we have to use another argument to prove something like $h^0(F(m))r' \geq h^0(F'(m))r$. This is exactly the content of a theorem of Le Potier. However, before we discuss it, we need a corollary of the Le Potier- Simpson estimates dicussed in section 4.

Lemma 7.5. [3, Corollary 3.3.8] Let $C = \frac{r(r+d)}{2}$. Then

$$h^0(F(m)) \leq \frac{r-1}{d!}[\mu'_{\max}(F) + C - 1 + m]_+^d + \frac{1}{d!}[\mu'(F) + C - 1 + m]_+^d.$$

We can now state the theorem of Le Potier we alluded to above:

Theorem 7.6. *Let p be a polynomial of degree d, and let r be a positive integer. Then for all sufficiently large integers m the following properties are equivalent for a purely $d-$dimensional sheaf F of multiplicity r and reduced polynomial p.*

(1) F is (semi)stable,

(2) $rp(m) \leq h^0(F(m))$, and $h^0(K(m)) \leq kp(m)$ for all subsheaves $K \subset F$ of multiplicity k, $0 < k < r$.

(3) $qp(m) \leq h^0(Q(m))$ for all quotient sheaves $F \twoheadrightarrow Q$ of multiplicity q, $0 < q < r$.

Proof. (1) $\implies$ (2): The idea is as follows: we know that the family of semistable sheaves with fixed Hilbert polynomial is bounded, but this fails for the family of subsheaves. However, we know, from Grothendieck's lemma, that the family of (certain) quotients with bounded from above slope is bounded, so the family of (saturated) subsheaves with bounded from below slope is bounded. Then, we will need a different argument for subsheaves with "small" slope.

As we said above, the family of semistable sheaves with Hilbert polynomial equal to rp is bounded by Theorem 4.4. Therefore, if m is sufficiently large, any such sheaf is $m-$regular, and $rp(m) = h^0(F(m))$. Let $K \subset F$ and let $C = \frac{r(r+d)}{2}$.

Case 1: $\mu'(K) < \mu'(F) - rC$. By lemma 7.4, we have that

$$h^0(K(m)) \leq \frac{k-1}{d!}[\mu'_{\max}(K) + C - 1 + m]^d_+ + \frac{1}{d!}[\mu'(K) + C - 1 + m]^d_+.$$

But we have $\mu'(K) < \mu'(F) - rC$ and $\mu'_{\max}(K) \leq \mu'(F)$ by the semistability of F, which together imply that

$$\frac{h^0(K(m))}{k} \leq \frac{m^d}{d!} + \frac{m^{d-1}}{(d-1)!}(\mu'(F) - 1) + \text{lower terms.}$$

Because $p(m) = \frac{m^d}{d!} + \frac{m^{d-1}}{(d-1)!}\mu'(F) + \text{lower terms}$, we deduce that

$$h^0(K(m)) \leq kp(m)$$

for sufficiently large m and all $K \subset F$ as above.

Case 2. $\mu'(K) \geq \mu'(F) - rC$.

The family of saturated sheaves $K \subset F$ is bounded by Grothendieck's lemma, where recall that a saturated sheaf $K \subset F$ is by definition one for which F/K is pure of dimension d equal to the dimension of the support of F. This means that there exists a large m such that all these sheaves K are

$m-$regular, implying that $h^0(K(m)) = P(K,m)$, and, moreover, that the set of Hilbert polynomials they can have is finite. We can choose m big enough such that

$$P(K,m) \leq kp(m) \leftrightarrow P(K) \leq kp,$$

and choosing m like this shows that (1) implies (2).

(2) $\implies$ (3) is immediate.

(3) $\implies$ (1): apply (3) to the maximal destabilizing quotient sheaf Q of F. Then, by Lemma 7.4,

$$p(m) \leq \frac{h^0(Q(m))}{q} \leq \frac{1}{d!}[\mu'(Q) + C - 1 + m]_+^d.$$

This shows that $\mu'_{\min}(F) = \mu'(Q)$ is bounded from below and consequently $\mu_{\max}(F)$ is bounded from above. Hence by Theorem 4.4 the family of sheaves F satisfying (3) is bounded. Now let Q be any purely $d-$dimensional quotient of F which destabilizes F, so $\mu'(F) \leq \mu(Q)$. Using Grothendieck's lemma, the family of such quotients Q is bounded, so we can choose m large enough such that $h^0(Q(m)) = P(Q,m)$ and

$$P(Q,m) \geq qp(m) \leftrightarrow P(Q) \geq qp,$$

which indeed show that (3) implies (1). □

After finding these equivalent characterizations of (semi)stable sheaves and (semi)stable points in GIT sense, we are ready to prove Theorem 6.2. In these notes/ talk, we only show that points corresponding to (semi)stable sheaves are GIT (semi)stable. For the full proof, see [3, Section 4.4].

Proof. Recall that the Hilbert polynomial $P = rp$ is fixed. Choose m big enough such that all families of semistable sheaves with Hilbert polynomial $p, 2p, ..., rp$ have regularity m. This can be done as each of these families is bounded.

Let $\rho : V \otimes \mathcal{O}(-m) \to F$ be a closed point in R. By definition of R, the map $V \to H^0(F(m))$ is an isomorphism. Let $F' \subset F$ be a subsheaf of multiplicity $0 < r' < r$ and let $V' = V \cap H^0(F'(m))$. Using Le Potier's theorem, we have either $p(F') = p(F)$ or $h^0(F'(m)) < r'p(m)$. In the first case F' is $m-$regular and we get $\dim(V') = h^0(F'(m)) = r'p(m)$ and therefore

$$\dim(V')P(F) = (r'p(m))(rp) = (rp(m))(r'p) = \dim(V)P(F').$$

In the second case

$$\dim(V)r' = rr'p(m) > h^0(F'(m)) = \dim(V)r.$$

These are the leading coefficients of $\dim(V)P(F')$ and $\dim(V')P(F)$, so indeed

$$\dim(V)P(F') > \dim(V')P(F).$$

By Lemma 7.3, we can deduce that (semi)stable sheaves correspond to (semi)stable GIT points. □

Before we finish, we should make an apology for not talking about the existence of universal family of stable sheaves. We have argued that the moduli functor Φ is not representable in general by looking at semistable sheaves with the same Harder-Narasimhan factors, but there might be hope that the moduli functor of stable sheaves Φ^s is representable. For example, if we write

$$P(n) = \sum_{i=0}^{d} a_i \binom{n+i-1}{i},$$

with integral coefficients $a_0, ..., a_d$, then $\gcd(a_0, ..., a_d) = 1$ implies that Φ^s is representable. For more details, see [3, Section 4.6].

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