Singularities of moduli spaces of sheaves on $K3$ surfaces and Nakajima quiver varieties, after Arbarello and Saccà

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Abstract

Following [1] very closely, we analyze the singularities of moduli spaces of sheaves on $K3$ surfaces for primitive Mukai vector. For a generic polarization, these moduli spaces are smooth, but this is not true for any polarization H. These singularities admit symplectic resolutions by choosing generic polarizations close to H. For pure sheaves of dimension one, these singularities are locally isomorphic to singularities of certain Nakajima quiver varieties, and the symplectic resolutions obtained by choosing a generic polarization close to H correspond to variations of GIT quotient in the construction of Nakajima quiver varieties.

1 Introduction

In these notes, we continue our study of the geometry of the moduli spaces of sheaves on $K3$ surfaces. Before we recall some of the results we have discussed in the seminar so far, we need to fix some notation. We will be concerned with the study of the moduli space $M_H(v)$ of (Gieseker semistable) sheaves F on a projective $K3$ surface S with Mukai vector

$$v = v(F) := \mathrm{ch}(F)\sqrt{\mathrm{td}(S)} = (r(F), c_1(F), \chi(F) - r(F)) \in H^*_{alg}(S, \mathbb{Z}),$$

where $r(F)$ is the rank of F, $c_1(F)$ the first Chern class, and $\chi(F)$ is the Euler characteristic of the sheaf F. Recall also the Mukai pairing for vectors $v = (v_0, v_1, v_2)$ and $w = (w_0, w_1, w_2)$ defined by $v \cdot w = v_1w_1 - v_0w_2 - v_2w_0$. Finally, $H \in \mathrm{Amp}\ (S)$ is a polarization in the ample cone of S.

In the second half of this seminar, we have discused some aspects of the geometry of the moduli space $M_H(v)$. First, we have seen that the locus of stable sheaves $M^s_H(V) \subset M_H(v)$ is smooth because the obstruction of this space being smooth at $[F]$ lies in $\mathrm{Ext}^2_0(F,F)=0$. There exists a symplectic form on $M^s_H(V)$, defined using the identification $T_{[F]}M^s_H(v)\cong \mathrm{Ext}^1(F,F)$ and the pairing

$$\mathrm{Ext}^1(F,F)\otimes \mathrm{Ext}^1(F,F)\to \mathrm{Ext}^2(F,F)\cong \mathrm{Ext}^0(F,F)^*\cong \mathbb{C}.$$

We have also seen that if $M_H(v)$ is non-empty, then $\dim M_H(v)=v^2+2$.

The next natural question is to determine those vectors v for which $M_H(v)\neq\emptyset$. Recall that a vector v is called primitive if it cannot be written as $v=mw$ for another Mukai vector w and for some integer $m\geq 2$. Further, a vector $v\in H^*_{alg}(S,\mathbb{Z})$ is positive if it is primitive, if $v^2+2\geq 0$, and if

$$\begin{aligned}\text{either } v_0&>0,\\ \text{or } v_0=0, v_1 \text{ is effective and } v_2&\neq 0,\\ \text{or } v_0=v_1=0 \text{ and } v_2&>0.\end{aligned}$$

Theorem 1.1. *(Yoshioka, [1, page 5]) Let v be a positive Mukai vector. Then for every ample polarization $H\in Amp(S)$ and for every $m\geq 1$, the moduli space $M_H(mv)$ is non-empty.*

In particular, if v is positive, we have that for generic H, the space $M_H(v)=M^s_H(v)$ (see section 2 for more details) is an irreducible holomorphic symplectic variety, which we will call in the rest of the notes IHS. It is natural to see what happens when v is not primitive or H is not generic.

We have already discussed the case when v is not primitive in this seminar. First, these moduli spaces are singular, but considering that in the primitive case they are smooth IHS varieties, it is natural to ask whether these singular spaces $M_H(mv)$, for $m\geq 2$, admit IHS resolutions. O'Grady found examples of such resolutions in some particular cases and he showed that these resolutions were not deformation equivalent to $M_H(v)$ for v primitive, as dicussed in Barbara's talk [2]. Kaledin, Lehn, and Sorger showed that the moduli spaces found by O'Grady are the only moduli spaces which admit a symplectic resolution of singularities. For more details, see Yinbang's talk [4].

The article [1] discusses the case when H_0 is not chosen generic, and v is a primitive Mukai vector. In this case, we will explain that choosing a generic polarization H near H_0 (in a sense to be made precise in the next section), we get a symplectic resolution

$$M_H(v)\to M_{H_0}(v),$$

which sends an $H-$stable sheaf F to its $S-$equivalence class with respect to H_0. Thus, this construction does not provide new examples of IHS. Nevertheless, it is interesting to study the singularities of the space $M_{H_0}(v)$

and see how the resolution of singularities $M_H(v) \to M_{H_0}(v)$ changes as H varies in a neighborhood of H_0 in the ample cone $\mathrm{Amp}(S)$. In these notes we will only be concerned with the case of pure one dimensional sheaves, when $v(F) = (0, c_1(F), \chi(F))$. In this situation, Gieseker semi-stability with respect to a polarization H is equivalent to slope stability for the slope $\mu_H(F) = \frac{\chi(F)}{c_1(F).H}$, because $\chi(mF) = (c_1(F).H)m + \chi(F)$.

Nakajima quiver varieties are symplectic varieties related to moduli of representations of quivers. Their construction bears some similarities with the construction of the moduli of sheaves, and their local geometry is easier to understand than that of moduli of sheaves, in general, as we will see in sections 3 and 4. The definition of Nakajima quiver varieties $\mathfrak{M}_{\chi_0}(\underline{n})$ depends on a quiver Q, a dimension vector $\underline{n}$, and a character χ_0 from a certain group to $\mathbb{C}^*$. As it is the case for moduli of sheaves, $\mathfrak{M}_{\chi_0}(\underline{n})$ is smooth for a generic choice of character, and the singular spaces $\mathfrak{M}_{\chi_0}(\underline{n})$ admit symplectic resolutions by choosing a generic χ near χ_0.

Arbarello and Saccà construct a quiver $Q(F)$ depending on a sheaf F such that the resolution

$$M_H(v) \to M_{H_0}(v)$$

near $[F]$ looks locally like a resolution of singularities

$$\mathfrak{M}_\chi(\underline{n}) \to \mathfrak{M}_{\chi_0}(\underline{n})$$

of Nakajima quiver varieties. Also, all the maps $M_H(V) \to M_{H_0}(v)$ obtained by varying the polarization $H \in \mathrm{Amp}(S)$ can be realized as maps of Nakajima quiver varieties associate to this quiver $Q(F)$. These results will be made precise in section 5.

The plan for these notes is the following: in section 2, we discuss the wall and chamber decomposition of $\mathrm{Amp}(S)$ and we explain how to construct resolutions of $M_{H_0}(v)$ for non-generic H_0 by "perturbing" H_0 in the ample cone. In section 3 we discuss a very important technical result about the deformation theory of $F \in M_{H_0}(v)$, namely that the dgla $R\operatorname{Hom}(F, F)$ satisfies the formality property. All these terms will be defined in the respective section, where we also explain the geometric consequences of this fact. In section 4 we give a brief overview of quivers, stability conditions, and Nakajima quiver varieties. Finally, in section 5 we state the main theorem from [1] and discuss some of the ingredients used in its proof.

2 Moduli of sheaves for non-generic polarizations

In this section we will explain how to construct (symplectic) resolutions of $M_{H_0}(v)$ for H_0 a not necessarily generic polarization. First, we need to know that the ample cone $\mathrm{Amp}(S)$ admits a wall-and-chamber decomposition:

Theorem 2.1. *(Yoshioka, see [1, Theorem-Definition 2.4])*

Let $\in H^*_{alg}(S,\mathbb{Z})$ be a positive (and thus primitive) Mukai vector. There exists a countable set of real codimension one linear subspaces in $Amp(S)\otimes\mathbb{R}$, called the walls associated to v, such that if H lies in the complement of these subspaces, then there are no strictly $H-$semistable sheaves with Mukai vector v, while if H lies on one of the walls, then there exist $H-$strictly semistable sheaves. If $W_1, ..., W_k$ are some walls, then $M_H(v)$ is independent of H a polarization lying precisely on these k walls. Finally, if v is the Mukai vector of a pure dimension one sheaf, then the number of walls is finite.

In the case of pure dimension one sheaves, one can describe fairly explicitly the walls inside $\mathrm{Amp}(S)$ using linear equations. A very similar description using linear equations exists in the wall-and-chamber decomposition for quiver varieties, as we will see in section 4. This description allows us to see that for H a polarization in a chamber adjacent to H_0, there exists a birational morphism

$$M_H(v) \to M_{H_0}(v).$$

Lemma 2.2. Let $v = (0, D, \chi)$ be a positive Mukai vector, with D is effective and $\chi \neq 0$. Then the walls associated to v are described by equations $\chi(\Gamma.x) = \chi_\Gamma(D.x)$, where $\Gamma \subset D$ is a subcurve, and χ_Γ ranges in a finite subset of $\mathbb{Z}$ determined by v and Γ.

Further, for any H adjacent to H_0, there exists a morphism

$$h : M_H(v) \to M_{H_0}(v)$$

which sends an $H-$stable sheaf F to the direct sum of its H_0-Jordan-Holder factors. The map is an isomorphism over the locus of H_0-stable sheaves, and it is birational.

For a proof, see [1, Proposition 2.5]. In the rest of the notes, we will always assume that F is polystable. This will cause no harm because in any $S-$equivalence class of sheaves there exists a polystable sheaf. In most of the cases, we will actually write $F = \oplus_{i=1}^{s} F_i \otimes V_i$ for F_i different stable sheaves and V_i complex vector spaces.

3 Deformation theory and the formality property

The general approach in deformation theory is to associate a deformation functor to a given deformation problem, and to try to represent this functor in some way. For example, when studying deformations of a sheaf, we are led to the functor D_F : Artin $\to$ Sets from the category of local Artinian $\mathbb{C}-$algebras to the category of sets, defined by

$$D_F(A) = \{(F_A, \phi) | F_A \text{ flat family of coherent sheaves on } S, \phi : F_A \otimes_A k \to F\} / \sim,$$

for the natural equivalence relation [1, page 9].

Now, if we could represent this functor by a space D_F in some sense, the moduli of sheaves around F will be a quotient of D_F by $\mathrm{Aut}(F)$. We will make precise statements involving this heuristics later. In any case, understanding this particular deformation functor gives information about the local geometry of the moduli of sheaves around F.

A general idea of the study of deformation problems is that, instead going straight from such a problem to a deformation functor, we should first find a differential graded lie algebra, also denoted by D_F, and then go from this dgla to the functor D_F. One reason for first constructing a dgla is that there exists a canonical way of associating a deformation functor to a dgla. Further, quasi-isomorphic dglas give isomorphic deformation functors. Let's explain the first statement in more detail.

First, recall that a dgla is a differential graded vector space (L, d) with a lie bracket $[\cdot, \cdot] : L \times L \to L$ of degree zero which is (graded) skew-symmetric, satisfies the (graded) Jacobi rule, and such that differentiating with respect to d respects a Leibniz rule$-$ for more details, see [5, 6]. Now, given a dgla L, we can associate a deformation functor MC_L : Artin $\to$ Sets which associates to a local Artinian ring solutions to the Maurer-Cartan equation:

$$MC_L(A) = \{x \in L^1 \otimes m_A | dx + \frac{1}{2}[x, x] = 0\}/\text{equivalence},$$

where m_A is the maximal ideal of A.

For our problem, it is natural to look for the dgla L which gives the deformation functor D_F needed in our study of moduli of sheaves. Fortunately, one can find it very explicitly: it is $R\,\mathrm{Hom}(F, F)$ [6]. We also say that F satisfies the formality criterion if $R\,\mathrm{Hom}(F, F)$ is quasi-isomorphic to its cohomology algebra $\mathrm{Ext}(F, F)$. If this is the case, the deformation functor D_F, which comes from the dgla $R\,\mathrm{Hom}(F, F)$, is the same as the deformation functor coming from the dgla (with trivial differentials) $\mathrm{Ext}(F, F)$. It is easier to study the deformation functor for such a dgla and one can show that it is represented by a complete intersection of quadrics in $\mathrm{Ext}^1(F,F)$. More precisely, the space D_F representing the deformation functor is isomorphic to $k_2^{-1}(0)$, where

$$k_2 : \widehat{\mathrm{Ext}^1(F,F)} \to \mathrm{Ext}^2(F,F)_0$$

is the Kuranishi map $k_2(e) = e \cup e$.

This result is very remarkable, because in general the space representing the deformation functor is $k^{-1}(0)$, where $k = k_2 + k_3 + \cdots$ is the full Kuranishi map, where k_n are the obstructions to lifting the sheaf to the level $n+1$, and they are the order n factors of the Kuranishi map. In our case $k = k_2$ is a quadratic map which has a very explicit description.

Thus, showing that $R\,\mathrm{Hom}(F,F)$ is formal gives a fairly explicit description of the moduli space $M_H(v)$ near $[F]$. Arbarello-Saccà showed, based on a theorem of Zhang [8], that all pure one dimensional sheaves satisfy the formality property.

Theorem 3.1. *Let F be a pure, dimension one sheaf on S which is polystable with respect to a polarization H_0. Then the dgla $R\,\mathrm{Hom}(F,F)$ is formal.*

The proof is based on a result of Zhang [8], who proved that $R\,\mathrm{Hom}(E,E)$ is formal for certain vector bundles E. Write $E = \oplus E_i^{n_i}$, where E_i are its non-isomorphic stable summands; then Zhang's result says that E is formal if all the E_is are line bundles or all have rank at least 2. We can assume that F is generated by global sections and that $H^1(S,F) = 0$ after tensoring F enough times with H_0. To a one dimensional sheaf F generated by global sections and with $H^1(S,F) = 0$, one can associate the Lazarsfeld-Mukai bundle M_F defined via the short exact sequence

$$0 \to M_F \to H^0(S,F) \otimes \mathcal{O}_S \to F \to 0.$$

One can show that M_F is locally free, and that E_F, the dual of M_F, satisfies the hypothesis of Zhang's theorem, and thus that the above result holds. Full details can be found in [1, Section 3].

4 Quivers and Nakajima quiver varieties

In this section, we finally define quivers, and talk about stability conditions, quiver varieties, and Nakajima quiver varieties. General references for the material in this section are [3] and [7].

First, a quiver Q is an oriented graph (V,E,s,t), where $s,t : E \to V$ are the source and target maps. We label the vertices by the numbers $\{1, 2, \cdots, s\}$.

A representation of dimension $\underline{n} = (n_1, \cdots, n_s)$ is a choice of maps $V_{s(e)} \to V_{t(e)}$, for every edge $e \in E$, where V_i is a complex vector space of dimension n_i. Thus, the space of representations of dimension $\underline{n}$ is

$$\mathrm{Rep}(Q, \underline{n}) = \oplus_{e \in E} \mathrm{Hom}(V_{s(e)}, V_{t(e)}),$$

on which the group

$$G(\underline{n}) := \prod_{i \in V} GL(n_i)$$

acts in the natural way. We want to define moduli spaces of representations as quotients of $\mathrm{Rep}(Q, \underline{n})$ by $G(\underline{n})$. Recalling the GIT setting discussed in lecture 3, we can define spaces

$$\mathrm{Rep}(Q, \underline{n})//_{\chi} G(\underline{n})$$

for all characters $\chi : G(\underline{n}) \to \mathbb{C}^*$. One can also define a stability conditions for quivers (called King-Rudakov), reminiscent of the slope/ Gieseker stability for moduli of sheaves.

To define it, let $\underline{n}$ be a dimension vector and let V be a representation of Q of dimension $\underline{n}$. Further, choose $\theta \in \underline{n}^{\perp} \subset \mathbb{Z}^s$. The representation V is called (semi)stable if for all proper subrepresentations $W \subset V$ of Q,

$$\frac{\sum \theta_i w_i}{\sum w_i} < (\leq) \frac{\sum \theta_i v_i}{\sum v_i}.$$

Recall that one of the most important ingredients in the construction of the moduli of sheaves was proving that Gieseker stability is the same as GIT stability (for some explicit character). There is an analogous statement for quivers. First, to a vector $\theta \in \underline{n}^{\perp}$, we associate the character $\chi_\theta : G(\underline{n}) \to \mathbb{C}^*$ by

$$\chi_\theta(g_1, ..., g_s) = \prod \det(g_i)^{-\theta_i}.$$

Theorem 4.1. *A representation V of dimension $\underline{n}$ of Q is King-Rudakov $\theta-$(semi)stable if and only if it is GIT (semi)stable for the character χ_θ. Also, if V' is another such representation, V and V' are King-Rudakov $S-$equivalent if and only if they are GIT $S-$equivalent.*

Before going further, we will mention some examples of quiver varieties. Let Q be the quiver with two vertices 1 and 2, and with r arrow $1 \to 2$. Choose the dimension vector $\underline{n} = (1, 1)$. The stability parameter will thus have the form $\theta = (u, -u)$. One can check that for $u < 0$ there are no semistable objects, so the moduli space is the empty set. Further, for $u = 0$ the quiver variety is a point, and for $u > 0$ the quiver variety if $\mathbb{P}^{r-1}$.

We did not mention anything about symplectic varieties yet. Given a smooth variety X, there is a standard way of constructing a symplectic variety: take the cotangent bundle T^*X. Thus, if we want to construct a symplectic variety from a quiver, we should take the cotangent bundle of $\mathrm{Rep}(Q,\underline{n})//_\chi G(\underline{n})-$ the problem is that the quotient may be very well singular. Nakajima quiver varieties are symplectic substitutes for the possibly absent cotangent bundle T^*X. The construction is based on the moment map, which is a very important tool in the study of moduli problems [3], but we will not discuss it in these notes.

Recall that

$$\mathrm{Rep}(Q,\underline{n}) = \oplus_{e\in E}\mathrm{Hom}(V_{s(e)},V_{t(e)}),$$

and thus

$$T^*\mathrm{Rep}(Q,\underline{n}) = \oplus_{e\in E}(\mathrm{Hom}(V_{s(e)},V_{t(e)})\oplus\mathrm{Hom}(V_{t(e)},V_{s(e)})).$$

The moment map $\mu : T^*\mathrm{Rep}(Q,\underline{n}) \to \mathfrak{g}(\underline{n})$ has in this particular case the explicit description:

$$\mu((x_e,y_e)_{e\in E}) = \sum_{e\in E}(x_e y_e - y_e x_e) \in \mathfrak{g}(\underline{n}),$$

where $\mathfrak{g}(\underline{n})$ is the lie algebra of $G(\underline{n})$. Given a character χ_θ as above, we can define a Nakajima quiver variety as the GIT quotient

$$M_\theta(\underline{n}) = \mu^{-1}(0)//_{\chi_\theta}G(\underline{n}).$$

One can put a symplectic structure on this variety [1, page 20].

In the case of moduli of sheaves, we were able to understand how the moduli spaces change in function of the polarization $H \in \mathrm{Amp}(S)$. There exists a wall-and-chamber structure in the case of quiver varieties as well:

Theorem 4.2. *(Nakajima) There exists a wall-and-chamber structure in $\underline{n}^\perp\otimes\mathbb{Q}\subset\mathbb{Q}^s$ with all the walls passing through the origin. Further,*

(1) *if θ is in a chamber, then $\theta-$semistability is the same as $\theta-$stability, so $M_\theta(\underline{n}) = M^s_\theta(\underline{n})$.*

(2) *if θ and θ' are contained in the walls $W_1,...,W_k$ and in no other walls, then $M_\theta(\underline{n}) = M_{\theta'}(\underline{n})$.*

(3) *let F and F' be faces such that $F'\subset\bar{F}$, with $\theta\in F,\theta\in F'$, then $\theta-$(semi)stability implies $\theta'-$(semi)stability. Since all the faces contain 0 in their closure, there exists a map*

$$\pi : M_\theta(\underline{n}) \to M_0(\underline{n})$$

for any $\theta\in\underline{n}^\perp$.

Before moving to the discussion of the main result of these notes, we mention an example of Nakajima quiver varieties. Let Q be the quiver with two vertices 1 and 2, with one edge $1 \to 2$ and with one loop $1 \to 1$. Choose the dimension vector $\underline{n} = (n, 1)$, and let $\chi := \det : G(\underline{n}) \to \mathbb{C}^*$. Then $M_\chi(\underline{n}) = \mathrm{Hilb}(\mathbb{C}^2, n)$ is the Hilbert scheme of n points in $\mathbb{C}^2$, and the map $M_\chi(\underline{n}) \to M_0(\underline{n})$ is the Hilbert-Chow morphism

$$\mathrm{Hilb}(\mathbb{C}^2, n) \to S^n(\mathbb{C}^2).$$

Full details and more examples can be found in [3] and [7].

5 The main theorem

In the introduction, we said that one of the main results of [1] is that locally the space $M_{H_0}(v)$ is isomorphic to an open subset of a Nakajima quiver variety. Before explaining why this is true, we first need to associate a quiver $Q = Q(F)$ to a polystable sheaf $F \in M_{H_0}(v)$. For this purpose, write

$$F = \oplus_{i=1}^s F_i \otimes V_i.$$

The moduli space $M_{H_0}(v)$ near $[F]$ "looks" like a quotient of $k_2^{-1}(0)$ by $\mathrm{Aut}(F)$, where recall that $k_2 : \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)$ is the order two Kuranishi map. Similarly, the Nakajima quiver variety for the trivial character looks like a quotient of the kernel of the moment map

$$T^*\mathrm{Rep}(Q, \underline{n}) \to \mathfrak{g}(\underline{n})$$

by $G(\underline{n})$. Thus, we need to construct a quiver Q such that

$$G(\underline{n}) \cong \mathrm{Aut}(F), \tag{5.1}$$
$$\mathrm{Rep}(Q, \underline{n}) \cong \mathrm{Ext}^1(F, F), \tag{5.2}$$
$$\mathfrak{g}(\underline{n}) \cong \mathrm{Ext}^2(F, F) \tag{5.3}$$

are all $G(\underline{n})-$equivariant isomorphisms. Having the explicit description of $F = \oplus_{i=1}^s F_i \otimes V_i$ we can compute

$$\begin{aligned} \mathrm{Aut}(F) &= \prod_{i=1}^s GL(V_i), \\ \mathrm{Ext}^1(F, F) &= \oplus_{i,j} \mathrm{Hom}(V_i, V_j)^{\mathrm{ext}^1(V_i, V_j)}, \\ \mathrm{Ext}^2(F, F) = \mathrm{Hom}(F, F) &= \oplus_{i=1}^s \mathrm{Hom}(V_i, V_i). \end{aligned}$$

Now, we can guess what is the right definition for the quiver Q. Label the vertices V by the integers $\{1, \cdots, s\}$ and draw

$$\begin{aligned} \mathrm{ext}^1(F_i, F_j) &\text{ for } i < j, \\ \mathrm{ext}^1(F_i, F_i)/2 &\text{ for } i = j, \\ 0 &\text{ for } j < i \end{aligned}$$

edges between the vertices i and j. One can show by direct computations now that (5.1), (5.2), and (5.3) all hold.

After all this preparation, we can finally state the main theorem from [1]:

Theorem 5.1. *Let H_0 be a polarization of S, and let $F_1, \cdots, F_s$ be pairwise distinct H_0-sheaves. Let $V_1, \cdots, V_s$ be vector spaces of dimension $n_1, \cdots, n_s$ respectively, and define $F := \oplus_{i=1}^s F_i \otimes V_i$ the corresponding H_0-polystable sheaf. Denote by v its Mukai vector and by $G = Aut(F) = \prod_{i=1}^s GL(V_i)$.*

(1) If F satisfies the formality criterion, then there exists a local analytic isomorphism

$$\phi : (\mathfrak{M}_0(\underline{n}), 0) \cong (M_{H_0(v)}, [F]).$$

(2) Suppose that F is pure of dimension one, so that in particular (1) *is satisfies. Then for any chamber $C \subset Amp(S)$ containing H_0 in its closure, we can find a chamber $D \subset \underline{n}^\perp$ such that for every $H \in C$ and for every $\theta \in D$, the symplectic resolution*

$$M_H(v) \to M_{H_0}(v)$$

is locally isomorphic to the symplectic resolution

$$\mathfrak{M}_\theta(\underline{n}) \to \mathfrak{M}_0(\underline{n}).$$

(3) The assignment of a chamber $C \subset \underline{n}^\perp \otimes \mathbb{Q}$ for every chamber in $Amp(S)$ which is adjacent to H_0 is induced as follows: if H is a polarization with $H.D = H_0.D$, then the morphism is

$$H \to \chi_H((g_1, \cdots, g_s)) = \prod_{i=1}^s \det(g_i)^{D_i.H - D_i.H_0},$$

where $D_i = c_1(F_i)$.

We will only discuss part (1) of the above theorem. We want to prove that $M_{H_0}(v)$ and $\mathfrak{M}_0(\underline{n}) = \mu^{-1}(0)//G(\underline{n})$ are locally isomorphic around $[F]$ and 0. Recall from lecture 4 that $M_{H_0}(v)$ is defined as a GIT quotient

$\mathrm{Quot}_{H_0}(W)//GL(W)$ and that we can find an étale slice $Z \subset \mathrm{Quot}_{H_0}(W)$ passing through $[F] = [\oplus_{i=1}^{s} F_i \otimes V_i]$ such that, for $G = \mathrm{Aut}(F)$,

$$Z//G \to \mathrm{Quot}_{H_0}(W)//GL(W) = M_{H_0}(v)$$

is étale. Now, we know that

$$\mu^{-1}(0) \cong k_2^{-1}(0)$$

$G(\underline{n})-$equivariantly, by our construction of the quiver Q. The formality property for F translates into a local $G(\underline{n})-$equivariant isomorphism

$$Z \cong k_2^{-1}(0).$$

This is explained in detail in [1, Section 4]. Thus, we have a local analytic isomorphism $Z//G(\underline{n}) \cong k_2^{-1}(0)//G(\underline{n}) = \mathfrak{M}_0(\underline{n})$. However, the map $Z//G(\underline{n}) \to M_{H_0}(v)$ is étale, and this implies the desired isomorphism

$$(M_{H_0}(v), [F]) \cong (\mathfrak{M}_0(\underline{n}), 0).$$

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