General introduction to K3 surfaces

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1 Algebraic K3 surfaces

1.1 Definition of K3 surfaces

Let $\mathbb{K}$ be an arbitrary field. Here, a *variety* over $\mathbb{K}$ will mean a separated, geometrically integral scheme of finite type over $\mathbb{K}$. A *surface* is a variety of dimension two. If X is a variety over $\mathbb{K}$ of dimension n, then ω_X will denote its canonical class, that is $\omega_X \cong \Omega^n_{X/\mathbb{K}}$. For a sheaf $\mathcal{F}$ on a scheme X, I will write $\mathrm{H}^\bullet(\mathcal{F})$ for $\mathrm{H}^\bullet(X, \mathcal{F})$, unless that leads to ambiguity.

Definition 1.1.1. A *K3 surface* over $\mathbb{K}$ is a complete non-singular surface X such that $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Corollary 1.1.1. One can observe several simple facts for a K3 surface:

1. $\Omega_X \cong \mathcal{T}_X$;
2. $\mathrm{H}^2(\mathcal{O}_X) \cong \mathrm{H}^0(\mathcal{O}_X)$;
3. $\chi(\mathcal{O}_X) = 2 \dim \Gamma(\mathcal{O}_X) = 2$.

Fact 1.1.2. Any smooth complete surface over an algebraically closed field is projective.

This fact is an immediate corollary of the Zariski–Goodman theorem which states that for any open affine U in a smooth complete surface X (over an algebraically closed field), the closed subset $X \setminus U$ is connected and of pure codimension one in X, and moreover supports an ample effective divisor. The proof of the theorem can be found in [2], Theorem 2.8.

Corollary 1.1.3. If $\mathbb{K}$ is algebraically closed, then a K3 surface over $\mathbb{K}$ is projective.

Remark 1.1.1. There is a notion of complex K3 surfaces. Those are not necessarily projective, but it is known that a complex K3 surface is projective if and only if it is algebraic.

Example 1.1.2 (Quartic in $\mathbb{P}^3$)**.** Let X be a smooth quartic in $\mathbb{P}^3$. Then one has a short exact sequence of coherent sheaves on $\mathbb{P}^3$:

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_X \to 0.$$

From the corresponding long exact sequence of cohomology one can easily derive that $\mathrm{H}^1(\mathcal{O}_X) = 0$. Now, from the adjunction formula, we can calculate the canonical bundle on X:

$$\omega_X \cong \big(\omega_{\mathbb{P}^3} \otimes \mathcal{O}(4)\big)|_X \cong \mathcal{O}_{\mathbb{P}^3}|_X \cong \mathcal{O}_X.$$

Hence X is a K3 surface.

Example 1.1.3. A smooth complete intersection of type $(d_1, \dots, d_n)$ in $\mathbb{P}^{n+2}$ is a K3 surface if and only if $\sum d_i = n + 3$. The argument is essentialy the same as in the previous example, iterated n times.

Note that without loss of generality, we can consider only intersections with all d_i being positive, so this gives us only finitely many possibilities, namely:

$$\begin{aligned} &n = 1, \text{ type } (4);\\ &n = 2, \text{ type } (2,3);\\ &n = 3, \text{ type } (2,2,2). \end{aligned}$$

So we have constructed K3 surfaces of degrees 4, 6 and 8.

Example 1.1.4 (Kummer surface)**.** Here, assume that $\mathbb{K}$ is algebraically closed and not of characteristic 2. Let A be an abelian surface over $\mathbb{K}$. The theory of abelian varieties implies that the natural involution $\iota : A \to A$, $x \mapsto -x$ has 16 fixed points. (For this fact, a reference would be [5], Chapter 6.) The quotient by this involution has only rational double point singularities, which can be resolved by blowing up in these points. The resulting surface can be also obtained by first blowing up and then taking the quotient, which results in the following diagram:

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d, "\pi"'] & A \arrow[d] \\ X \arrow[r] & A/\iota \end{tikzcd}$$

The canonical bundle formula for blowup (see [3], V.Prop.3.3) implies that $\omega_{\tilde{A}} \cong \mathcal{O}(\sum E_i)$, where $\mathcal{E}_i$ are the exceptional divisors corresponding to the points. Note that π is a branched covering of degree 2, so if $\bar{E}_i$ are the images of the divisors E_i, then $\pi^*\mathcal{O}(\bar{E}_i) = \mathcal{O}(2E_i)$. Also, by the canonical bundle formula for branched coverings, we get $\omega_{\tilde{A}} \cong \pi^*\omega_X \otimes \mathcal{O}(\sum E_i)$. By comparing these two formulas, we obtain that $\pi^*\omega_X \cong \mathcal{O}_{\tilde{A}}$. Let $\mathcal{L}$ be a square root of $\mathcal{O}(\sum \bar{E}_i)$, then use the isomorphism $\pi_*\mathcal{O}_{\tilde{A}} \cong \mathcal{O}_X \oplus \mathcal{L}^*$ and the projection formula to conclude that the canonical bundle of X is trivial:

$$\mathcal{O}_X \oplus \mathcal{L}^* \cong \pi_*\mathcal{O}_{\tilde{A}} \cong \pi_*\pi^*\omega_X \cong \pi_*\mathcal{O}_{\tilde{A}} \otimes \omega_X \cong (\mathcal{O}_X \oplus \mathcal{L}^*) \otimes \omega_X \cong \omega_X \oplus (\mathcal{L}^* \otimes \omega_X)\,.$$

Now we can take canonical morphisms $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{L}^* \cong \omega_X \oplus (\mathcal{L}^* \otimes \omega_X) \to \omega_X$ and $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{L}^* \cong \omega_X \oplus (\mathcal{L}^* \otimes \omega_X) \to \mathcal{L}^* \otimes \omega_X$. One of them should be an isomorphism. In the former case, we get that $\mathcal{O}_X \cong \omega_X$, and in the latter case, we would obtain a contradiction.

Finally, note that the image of the injection $\mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(\tilde{A}, \mathcal{O}_{\tilde{A}}) \cong \mathrm{H}^1(A, \mathcal{O}_A)$ is contained in the subspace invariant under the induced action of ι, hence $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

For a more detailed discussion of Kummer surfaces, see [2], Chapter 10, Section 10.5.

Example 1.1.5 (Double plane)**.** Assume that $\mathbb{K}$ is not of characteristic 2. Consider a nonsingular curve $C \subset \mathbb{P}^2$ of degree six. Take line bundle $\mathcal{L} \cong \mathcal{O}(3)$ which is a square root of $\mathcal{O}(C)$, i.e. has a fixed isomorphism $\mathcal{L}^2 \cong \mathcal{O}(C)$, and a section $\sigma \in \Gamma(\mathcal{L}^2)$. Then we can consider the double covering $\pi : X \to \mathbb{P}^2$ branched along the curve C, defined as a hypersurface in $\mathrm{Tot}(\mathcal{L}^*)$ by local equation $t^2 = \sigma$.

Then $\pi_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-3)$, which implies that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Now use the canonical bundle formula for branched coverings:

$$\omega_X \cong \pi^*\left(\omega_{\mathbb{P}^2} \otimes \mathcal{O}(3)\right) \cong \pi^*\mathcal{O}_{\mathbb{P}^2} \cong \mathcal{O}_X.$$

This concludes the proof that X is a K3 surface.

1.2 Classical invariants

Intersection form. Hartshorne introduces intersections form on a surface in [3], Chapter V, §1, for (smooth projective) surfaces over algebraically close fields; moreover, other aspects of the classical theory are also defined over an algebraically closed field; so from now on we will assume that the base field $\mathbb{K}$ is algebraically closed. However, there is a formula for the intersection form in terms of Euler characteristic or Hilbert polynomials, which allows one to generalize the definition of the intersection form to the not algebraically closed case, but it seems excessive now.

Recall that by $\operatorname{Div} X$ we denote the group of divisors, and there is no ambiguity in this notion if X is a smooth projective variety. Recall also that if D is a divisor, then we can associate a line bundle to it, and this line bundle is denoted by $\mathcal{O}_X(D)$.

Theorem 1.2.1. Let X be a smooth projective surface. Then there is a unique pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$, denoted by $C.D$ for any two divisors C, D, such that

1. if C and D are nonsingular curves meeting transversally, then $C.D = \#(C \cap D)$;
2. the pairing is symmetric;
3. the pairing is bilinear, i.e. for any triple of divisors C_1, C_2 and D, we have $(C_1 + C_2).D = C_1.D + C_2.D$;
4. the pairing depends only on the linear equivalence classes, i.e. if $C_1 \sim C_2$, then $C_1.D = C_2.D$.

Definition 1.2.1. The pairing in the previous theorem is called the *intersection form* of the surface.

Fact 1.2.2. Now we can list without proof several facts about the intersection form. First note that for a smooth surface X, we have a surjection $\operatorname{Div}(X) \to \operatorname{Pic}(X)$ with kernel being divisors linearly equivalent to zero, so the intersection form induces a pairing on $\operatorname{Pic}(X)$.

1. if $\mathcal{L}$ is an ample line bundle and $C \subset X$ is a curve, then $\mathcal{L}.\mathcal{O}(C) > 0$;
2. Riemann–Roch for line bundles on surfaces:
$$\chi(\mathcal{L}) = \frac{\mathcal{L}.(\mathcal{L} \otimes \omega_X^*)}{2} + \chi(\mathcal{O}_X).$$

Corollary 1.2.3. If X is a K3 surface, then the Riemann–Roch theorem has a very nice form:
$$\chi(\mathcal{L}) = \frac{1}{2}\mathcal{L}.\mathcal{L} + 2.$$

Proof. Combine Corollary 1.1.1 and triviality of the canonical bundle. □

Definition 1.2.2. We define *algebraic equivalence* of divisors on X as the transitive closure of the relation $\sim$. For two effective divisors C, D in $\operatorname{Div}(X)$, we say that $C \sim D$ if there exist a nonsingular curve T and an effective divisor E on $X \times T$ flat (as a scheme) over T, and for some points $0, 1 \in T$, we have $E_0 \cong C$ and $E_1 \cong D$. For two arbitrary divisors C, D in $\operatorname{Div}(X)$, we say that $C \sim D$ if there exist effective divisors C_1, C_2, D_1, D_2 such that $C_i \sim D_i$ and $C = C_1 - C_2$, $D = D_1 - D_2$.

Definition 1.2.3. The *Néron–Severi group* of a surface X is the quotient $\mathrm{NS}(X) \stackrel{\mathrm{def}}{=} \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, where $\mathrm{Pic}^0(X)$ is the subgroup of all line bundles that are algebraically equivalent to zero.

Definition 1.2.4. The group $\mathrm{Num}(X)$ is defined as the quotient of the Picard group $\mathrm{Pic}(X)$ by the kernel of the intersection form. We say that line bundles from the kernel are *numerically trivial.*

Fact 1.2.4. The groups $\mathrm{NS}(X)$ and $\mathrm{Num}(X)$ are finitely generated.

Definition 1.2.5. The rank of $\mathrm{NS}(X)$ is called the *Picard number* and is denoted by $\rho(X)$.

Lemma 1.2.5. Let H be an ample divisor on a surface X. Then there is an integer n_0 such that for any divisor D, if $D.H > n_0$, then $\mathrm{H}^2(\mathcal{O}(D)) = 0$.

Proof. By Serre duality, $h^2(\mathcal{O}(D)) = h^0(\mathcal{O}(K - D))$, where K stands for a divisor corresponding to the canonical bundle ω_X. We want the divisor $K - D$ to be not effective. For this, it is sufficient that $(K - D).H < 0$, for H is ample. So we can take $n_0 = K.H$. $\square$

Lemma 1.2.6. Let H be an ample divisor on a surface X, and let D be a divisor such that $D.H > 0$ and $D^2 > 0$. Then for all $n >> 0$, nD is linearly equivalent to an effective divisor.

Proof. We apply the Riemann–Roch theorem to nD (here K stands for a divisor corresponding to the canonical bundle ω_X):

$$\chi(\mathcal{O}(nD)) = \frac{nD.(nD - K)}{2} + \chi(\mathcal{O}_X).$$

Apply the previous lemma to get that for n large enough, $h^2(\mathcal{O}(nD)) = 0$, so that $\chi(\mathcal{O}(nD)) = h^0(\mathcal{O}(nD)) - h^1(\mathcal{O}(nD)) \leq h^0(\mathcal{O}(nD))$. Apply these considerations to the Riemann–Roch formula:

$$h^0(\mathcal{O}(nD)) \geq \frac{nD.(nD - K)}{2} + \chi(\mathcal{O}_X).$$

But the right hand side is a polynomial of degree two in n and tends to infinity for large n, in particular $h^0(\mathcal{O}(nD))$ is positive for large n and hence nD is effective. $\square$

Theorem 1.2.7 (Hodge index theorem)**.** Let X be a surface, let H be an ample divisor on the surface, and suppose that D is a divisor which is not numerically trivial and such that $D.H = 0$. Then $D^2 < 0$.

Proof. We will prove by contradiction. Assume that $D^2 \geq 0$ and consider first the case when $D^2 > 0$. Let $H' = D + nH$. For n big enough, H' is an ample divisor, because H is ample. The inequality $D.H' = D^2 > 0$ implies, by the previous lemma, that mD is an effective divisor for large enough m, which would imply that $mD.H > 0$, which in turn implies that $D.H > 0$. That is a contradiction.

If $D^2 = 0$, then by assumption there is a divisor E with $D.E \neq 0$. Take $E' \stackrel{\text{def}}{=} (H^2)E - (E.H)H$, then

$$D.E' = (H^2)D.E \neq 0 \text{ and } E'.H = 0.$$

Now let $D' = nD + E'$, then $D'.H = 0$ and $D'^2 = 2nD.E' + E'.E'$. Since $D.E' \neq 0$, by a suitable choice of n we can make $D'^2 > 0$, and apply the previous argument to D' to get a contradiction. □

Proposition 1.2.8. Let X be a surface with an ample divisor, say H. Then the signature of the intersection form on $\mathrm{Num}(X)$ is $(1, \rho(X) - 1)$.

Proof. All nonzero elements of the orthogonal complement to H are squared to a negative number, by Hodge index theorem 1.2.7. □

Proposition 1.2.9. For a K3 surface X, the natural surjections are isomorphisms:

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{Num}(X).$$

In addition, the intersection form on $\mathrm{Pic}(X)$ is non-degenerate, even and of signature $(1, \rho(X) - 1)$.

Proof. Suppose that a line bundle $\mathcal{L}$ is non-trivial, but for some ample line bundle $\mathcal{L}'$ the pairing $\mathcal{L}.\mathcal{L}'$ is equal to zero. Then also $\mathcal{L}^*.\mathcal{L}' = 0$. Use Fact 1.2.2 to conclude that $\mathcal{L}$ and $\mathcal{L}^*$ do not arise as bundles corresponding to effective divisors in X, so they cannot possibly have nonzero global sections, i.e. $\mathrm{H}^0(\mathcal{L}) = \mathrm{H}^0(\mathcal{L}^*) = 0$. Now use Serre duality for $\mathrm{H}^2(\mathcal{L}) \cong \mathrm{H}^0(\mathcal{L}^*)^* = 0$. That means that $\chi(\mathcal{L}) = -h^1(\mathcal{L}) \leq 0$. Now apply Riemann–Roch theorem for K3 surfaces:

$$\chi(\mathcal{L}) = \frac{1}{2}\mathcal{L}.\mathcal{L} + 2 \leq 0.$$

The latter implies that $\mathcal{L}.\mathcal{L} < 0$, in particular, $\mathcal{L}$ is not numerically trivial. That means that the surjections in consideration, namely

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{Num}(X),$$

are injective, i.e. they are isomorphisms.

The intersection form is non-degenerate on $\mathrm{Num}(X)$, so it is non-degenerate on $\mathrm{Pic}(X)$. Apply Riemann–Roch once again to obtain the formula $\mathcal{L}.\mathcal{L} = 2(\chi(\mathcal{L}) - 4)$ and conclude that the form is even. Claim about the signature follows from the Hodge index theorem 1.2.7. □

Chern classes. A reference for this background material is appendix A in [3], where Hartshorne defines Chow ring and Chern classes of an algebraic variety and states the Hirzebruch–Riemann–Roch formula.

Chern classes of vector bundles are naturally defined as elements of the Chow ring. The latter is freely generated, as an abelian group, by closed irreducible subvarieties in a variety X that are subject to a certain equivalence relation, called "rational equivalence". It is not essential to give here the definition of the rational equivalence, because we will only use formal properties of Chern classes without proof. Moreover, the Chow ring is graded by the codimension of the subvarieties. The ith homogeneous component of the Chow ring of a variety X is denoted by $A^i(X)$. Note that $A^0(X) \cong \mathbb{Z}$ naturally (because we assume that X is itself irreducible). This isomorphism is sometimes called deg, and we will occasionally speak of an element $a \in A^0(X)$ as of an integer.

Definition 1.2.6. Let $\mathcal{E}$ be a locally free sheaf of rank r on a nonsingular quasiprojective variety X. Let $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the natural projection. Then π^* makes $A(\mathbb{P}(\mathcal{E}))$ into a free $A(X)$-module generated by 1, ξ, ..., ξ^{r-1}. For each i =0, 1, ..., r, define the ith *Chern class* $c_i(\mathcal{E}) \in A^i(X)$ by the requirement $c_0(\mathcal{E}) = 1$ and

$$\sum_{i=0}^{r}(-1)^i \left(\pi^* c_i(\mathcal{E}).\xi^{r-i}\right) = 0.$$

Define the *total Chern class*

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E})$$

and the *Chern polynomial*

$$c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r.$$

Fact 1.2.10. Here we list some properties of Chern polynomials. Let X be a variety, let $\mathcal{E}'$, $\mathcal{E}$, $\mathcal{E}'$", $\mathcal{F}$ be locally free sheaves on X.

1. If $\mathcal{L} \cong \mathcal{O}_X(D)$ for a divisor D, then $c_t(\mathcal{L}) = 1 + Dt$. In particular, $c_t(\mathcal{O}_X) = 1$.

2. If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free sheaves on X, then $c_t(\mathcal{E}) = c_t(\mathcal{E}')c_t(\mathcal{E}'')$.

3. *(Splitting principle.)* The Chern polynomial of a locally free sheaf can be written down as a product
$$c_t(\mathcal{E}) = \prod_{i=1}^{\operatorname{rk}\mathcal{E}}(1 + a_i t),$$
where $a_i \in A^1(X)$.

4. For a vector bundle $\mathcal{E}$, $c_t(\det \mathcal{E}) = 1 + c_1(\mathcal{E})t$.

Hirzebruch–Riemann–Roch theorem. For the purpose of stating this theorem, we will need to introduce two elements of $A(X) \otimes \mathbb{Q}$ corresponding to a sheaf $\mathcal{E}$ of rank r on X. Let

$$c_t(\mathcal{E}) = \prod_{i=1}^{r}(1 + a_i t).$$

Definition 1.2.7. Define the *exponential Chern character* of $\mathcal{E}$ as

$$\operatorname{ch}(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i}.$$

Definition 1.2.8. Define the *Todd class* of $\mathcal{E}$ as

$$\operatorname{td}(\mathcal{E}) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}}.$$

Theorem 1.2.11 (Hirzebruch–Riemann–Roch)**.** For a locally free sheaf $\mathcal{E}$ on a nonsingular projective variety X of dimension n over an algebraically closed field, the following formula holds:

$$\chi(\mathcal{E}) = \deg\left(\operatorname{ch}(\mathcal{E}).\operatorname{td}(\mathcal{T})\right)_n,$$

where $\mathcal{T}$ is the tangent bundle on X and $(\,\cdot\,)_n$ denotes the component of degree n in $A(X) \otimes \mathbb{Q}$.

Hodge diamond of a K3 surface. We will now explain how to use Hirzebruch–Riemann–Roch formula to determine the Hodge numbers of a K3 surface X:

$$h^{p,q}(X) \stackrel{\text{def}}{=} \dim \mathrm{H}^q\left(\Omega_X^p\right).$$

Hodge numbers are usually written in the form of diamond, called *Hodge diamond*, which for surfaces looks as follows:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Interestingly, all K3 surfaces have the same Hodge diamond. We will now proceed by determining its entries.

Proposition 1.2.12. The Hodge diamond of a K3 surface X looks as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof. Recall that for a sheaf $\mathcal{E}$ of rank r on X we have:

$$\mathrm{ch}(\mathcal{E}) = r + c_1(\mathcal{E}) + \frac{1}{2}\left(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right)$$

and

$$\mathrm{td}(\mathcal{E}) = 1 + \frac{1}{2}c_1(\mathcal{E}) + \frac{1}{12}\left(c_1(\mathcal{E})^2 + c_2(\mathcal{E})\right).$$

First, apply the Hirzebruch–Riemann–Roch theorem 1.2.11 for $\mathcal{E} = \mathcal{O}$, $\mathcal{O}$ being the structure sheaf $\mathcal{O} = \mathcal{O}_X$. By 1.2.10, $c_t(\mathcal{O}) = 1$, so $\mathrm{ch}(\mathcal{O}) = 1$ and

$$\chi(\mathcal{O}) = \deg\left(1.\left(1 + \frac{1}{2}c_1(\mathcal{T}) + \frac{1}{12}\left(c_1(\mathcal{T})^2 + c_2(\mathcal{T})\right)\right)\right)_2 = \frac{1}{12}\left(c_1(\mathcal{T})^2 + c_2(\mathcal{T})\right).$$

Recall that $c_1(\mathcal{T}) = c_1(\det \mathcal{T}) = c_1(\mathcal{O}) = 0$, and use the fact that $\chi(\mathcal{O}) = 2$ for a K3 surface. So we get the following equality:

$$2 = \chi(\mathcal{O}) = \frac{c_2(\mathcal{T})}{12},$$

hence $c_2(\mathcal{T}) = 24$.

Now apply the Hirzebruch–Riemann–Roch formula to $\mathcal{T}$:

$$\chi(\mathcal{T}) = \deg\left(\mathrm{ch}(\mathcal{T}).\mathrm{td}(\mathcal{T})\right)_2 = \deg\left((2 - c_2(\mathcal{T})).\left(1 + \frac{c_2(\mathcal{T})}{12}\right)\right)_2 = 4 - 24 = -20.$$

Now turn to calculating the Hodge numbers. We will use the fact that in characteristic zero, Hodge diamond has a vertical and a horizontal symmetries. So, knowing that $h^0(\mathcal{O}) = 1$, we can conclude that $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$. Also, from $h^1(\mathcal{O}) = 0$ we get that $h^{1,0} = h^{0,1} = h^{1,3} = h^{3,1} = 0$. So we are left with only one unknown Hodge number $h^{1,1}$. But recall that $h^{1,1} = h^1(\Omega_X) = h^0(\Omega_X) + h^2(\Omega_X) - \chi(\Omega_X) = -\chi(\Omega_X) = -\chi(\mathcal{T}) = 20$. □

2 Complex K3 surfaces

2.1 Complex K3 surfaces

Definition 2.1.1. A *complex K3 surface* is a compact connected Kähler complex surface X such that $\Omega_X^2 \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Remark 2.1.1. If you drop the condition of a complex K3 surfaces being Kähler, it would still turn out that the surface is Kähler. The proof is highly nontrivial, so we just add this condition as a part of the definition. For details, see Section 7.3.2 in [4].

Proposition 2.1.1. If X is an algebraic K3 surface over $\mathbb{K} = \mathbb{C}$, then the associated complex space X^{an} is a complex K3 surface. Moreover, if a complex K3 surface is projective, then it can be obtained from an algebraic K3 surface.

To prove the proposition, one should recall the principles of GAGA. In particular, one wants to know that for projective varieties, categories of coherent sheaves of an algebraic variety and of the corresponding complex manifold are equivalent.

Example 2.1.2. We can generalize the notion of Kummer K3 surfaces to not necessarily algebraic complex tori. This will give an example of a not projective complex K3 surface. One can give an analytic proof of triviality of the canonical bundle. Recall the construction of a Kummer surface — A was a two-dimensional torus, then we have taken a quotient by an involution and blown up the singularities.

$$\begin{tikzcd}
\tilde{A} \arrow[r] \arrow[d, "\pi"'] & A \arrow[d] \\
X \arrow[r] & A/\iota
\end{tikzcd}$$

The manifold A trivially possesses a holomorphic symplectic form, which is preserved by the involution, so it induces a symplectic form on A/ι away from the singular locus. The idea is then to take a local analytic neighborhood of a singular fiber in X and use the fact that it is isomorphic to $T^*\mathbb{P}^2$.

Exponential short exact sequence. For a complex manifold, we have the following short exact sequence of sheaves:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0.$$

It yields the long exact sequence of cohomology for the case of K3 surfaces (note that there are nonzero terms further on the right):

$$\begin{gathered}
0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\sim} \mathbb{C} \to \mathrm{H}^1(X,\mathbb{Z}) \to 0 \to \mathrm{H}^1(\mathcal{O}^\times) \to \\
\to \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{C} \to \mathrm{H}^2(\mathcal{O}^\times) \to \mathrm{H}^3(X,\mathbb{Z}) \to 0.
\end{gathered}$$

This long exact sequence shows that $\mathrm{H}^1(X,\mathbb{Z}) = 0$, so we can write a new version of this long exact sequence (recall that $\mathrm{H}^1(\mathcal{O}^\times) \cong \mathrm{Pic}(X)$):

$$0 \to \mathrm{Pic}(X) \to \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{C} \to \mathrm{H}^2(\mathcal{O}^\times) \to \mathrm{H}^3(X,\mathbb{Z}) \to 0.$$

We already know that $\mathrm{Pic}(X)$ is torsion free, so $\mathrm{H}^2(X, \mathbb{Z})$ is also torsion free. By Poincaré duality and because $\mathrm{H}^2(X, \mathbb{Z})$ has zero torsion, we can conclude that $\mathrm{H}^3(X, \mathbb{Z}) = 0$. So we have only three nonzero cohomology groups for a K3 surface.

Picard group and topological interpretation of intersection form. The Picard group of a K3 surface X is embedded into $\mathrm{H}^2(X, \mathbb{Z})$ and hence inherits the topological intersection form. This induced intersection form coincides with the algebraically defined intersection form. This inclusion shows that the isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X)$ holds in the complex case as well.

Remark 2.1.3. However, for non-projective complex K3 surfaces it can happen that $\mathrm{Pic}(X)$ and $\mathrm{Num}(X)$ are not isomorphic.

One can give an upper bound on the rank of the Picard group. The Lefschetz theorem on (1,1)-classes states the following isomorphism (in fact, the theorem claims that it is true for any compact Kähler manifold X):

$$\mathrm{Pic}(X) \cong \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X).$$

The Hodge diamond of a complex K3 surface is the same as the one of an algebraic K3 surface, so we get the following inequality:

$$\mathrm{rk}\,\mathrm{Pic}(X) \cong \mathrm{rk}\,\left(\mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X)\right) \leq \dim \mathrm{H}^{1,1}(X) = 20.$$

In fact, any number between 0 and 20 is attained as the Picard number of a complex K3 surface. In the case of algebraic K3 surfaces over arbitrary base field, there is a lower bound $1 \leq \rho$, and by means of étale cohomology, one can find an upper bound $\rho \leq 22$.

Fact 2.1.2. The integral cohomology $\mathrm{H}^2(X, \mathbb{Z})$ of a complex K3 surface X endowed with the intersection form is isomorphic to the lattice

$$\mathrm{H}^2(X, \mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}.$$

The proof of this fact invokes theory of lattices, namely general classification of unimodular lattices. Due to this classification, it is enough to prove that $\mathrm{H}^2(X, \mathbb{Z})$ is even of signature $(3, 19)$. After that, a tedious (but not overwhelmingly) calculation would follow, which is done by Huybrechts in [4], Chapter 1, Proposition 3.5.

Notation 2.1.2. We will denote the lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ by Λ and call it the *K3 lattice*.

2.2 Hodge structures

Let V be a free $\mathbb{Z}$-module of finite rank or a finite-dimensional $\mathbb{Q}$-vector space. By $V_{\mathbb{C}}$ we will denote the vector space $V \otimes \mathbb{C}$ obtained by extension of scalars (tensor product is over $\mathbb{Z}$ or $\mathbb{Q}$, respectively). The complex vector space $V_{\mathbb{C}}$ naturally comes with a real structure, i.e. we have an $\mathbb{R}$-linear isomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$ defined by complex conjugation.

Definition 2.2.1. A *Hodge structure* of weight $n \in \mathbb{Z}$ on V is given by a direct sum decomposition of the complex vector space $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

For a Hodge structure V of even weight $n = 2k$ the intersection $V \cap V^{k,k}$ (defined by means of the natural inclusion $V \subset V_{\mathbb{C}}$) is called the space of *Hodge classes* in V.

Example 2.2.1. The *Tate Hodge structure* is denoted by $\mathbb{Z}(1)$. It is the Hodge structure of weight -2 given by the free $\mathbb{Z}$-module of rank 1 such that $(\mathbb{Z}(1))^{-1,-1}$ is one-dimensional. Similarly, one defines the rational Hodge structure $\mathbb{Q}(1)$. We can also define twists of the Tate Hodge structures $\mathbb{Z}(n)$ and $\mathbb{Q}(n)$ in an obviuos way.

Example 2.2.2. For a compact Kähler manifold (in particular, for a smooth complex projective variety), the torsion free part of the singular cohomology $\mathrm{H}^n(X, \mathbb{Z})$ comes with a natural Hodge structure of weight n given by the standard Hodge decomposition.

Definition 2.2.2. Define a *morphism* of Hodge structures V, W of weight n as a $\mathbb{Z}$-linear or $\mathbb{Q}$-linear, respectively, homomorphism $f : V \to W$ such that its $\mathbb{C}$-linear extension maps $V^{p,q}$ to $W^{p,q}$.

Definition 2.2.3. For the purpose of this definition, first introduce the notion of the *Weil operator* C, which acts on $V^{p,q}$ by multiplication with i^{p-q} (here i is a square root of unity). A *polarization* of a rational Hodge structure V of weight n is a morphism of Hodge structures

$$\psi : V \otimes V \to \mathbb{Q}(-n)$$

such that its $\mathbb{R}$-linear extension yields a positive definite symmetric form

$$(v, w) \mapsto \psi(v, Cw)$$

on the real part of $V^{p,q} \oplus V^{q,p}$. Then the data (V, ψ) is called a *polarized Hodge structure*. A Hodge structure is called *polarizable* if it admits a polarization. An isomorphism $V_1 \to V_2$ of Hodge structures that is compatible with given polarizations ψ_1, respectively ψ_2, is called a *Hodge isometry*.

2.3 Period map

Recall that Λ denotes the K3 lattice. Consider the $\mathbb{C}$-vector space $\Lambda_{\mathbb{C}} \overset{\text{def}}{=} \Lambda \otimes \mathbb{C}$ endowed with the $\mathbb{C}$-linear extension of the form on Λ. This extended form corresponds to a homogeneous quadratic polynomial. The latter defines a quadric in $\mathbb{P}(\Lambda_{\mathbb{C}})$, which is smooth because the bilinear form is nondegenerate.

Definition 2.3.1. We define the *period domain* as the following open (in classical topology) subset of the quadric:

$$D \stackrel{\text{def}}{=} \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,|\, (x, x) = 0,\ (x, \bar{x}) > 0\} \subset \mathbb{P}(\Lambda_{\mathbb{C}}).$$

Definition 2.3.2. We say that a Hodge structure V *is of K3 type* if $\dim V^{2,0} = \dim V^{0,2} = 1$, $\dim V^{1,1} = 20$ and all other spaces in the decomposition are zero.

Proposition 2.3.1. There exists a natural bijection between D and the set of Hodge structures of K3 type on Λ which for any $(2, 0)$-class σ satisfy:

1. $(\sigma, \sigma) = 0$;
2. $(\sigma, \bar{\sigma}) > 0$;
3. $\Lambda^{1,1} \perp \sigma$.

Proof. The $(2, 0)$ part of any Hodge structure of K3 type defines a line in $\Lambda_{\mathbb{C}}$, and conditions 1 and 2 guarantee that this line lies in D.

Conversely, if a point $l \in D$ is given, then there exists a Hodge structure with l as its $(2, 0)$ part, and this Hodge structure satisfies 1 and 2. The third condition defines the $(1, 1)$ part uniquely, so the Hodge structure satisfying conditions 1—3 is unique. □

Now consider a smooth proper family of K3 surfaces $f : X \to S$. Assume that S is connected and simply connected, with a distinguished point 0 and a fixed isomorphism $\varphi : \mathrm{H}^2(X_0, \mathbb{Z}) \to \Lambda$. As S is simply connected, we can assume that all fibres $\mathrm{H}^2(X_t, \mathbb{Z})$ are canonically isomorphic to Λ: $\mathrm{H}^2(X_t, \mathbb{Z}) \cong \mathrm{H}^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda$.

Fact 2.3.2. The *period map* defined by

$$\begin{aligned} \mathcal{P} : S &\to \mathbb{P}(\Lambda_{\mathbb{C}}) \\ t &\mapsto [\varphi(\mathrm{H}^{2,0}(X_t))] \end{aligned}$$

is a holomorphic map that takes values in the period domain $D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$. It depends on the distinguished point $0 \in S$ and the chosen isomorphism φ.

Definition 2.3.3. A smooth proper family $f : X \to S$ with a distinguished point $0 \in S$ is called the *universal deformation* if for any other smooth proper family $f' : X' \to S'$ with $X_0 \cong X'_0$, the latter family can be obtained from the first one as a pullback under a unique map $S' \to S$.

Fact 2.3.3. Let X_0 be a complex K3 surface. Then X_0 admits a smooth universal deformation $X \to \mathrm{Def}(X_0)$ with $\mathrm{Def}(X_0)$ smooth of dimension 20.

Theorem 2.3.4 (Local Torelli theorem)**.** Let X_0 be a complex K3 surface and consider its universal deformation $X \to \mathrm{Def}(X_0)$. Take a connected simply connected open neighborhood of 0 in $\mathrm{Def}(X_0)$, denote it by S. Then the period map

$$\mathcal{P} : S \to D \subset \mathbb{P}(\mathrm{H}^2(X_0, \mathbb{C}))$$

is a local isomorphism.

Fact 2.3.5. The universal families of K3 surfaces $X \to \mathrm{Def}(X_0)$ glue to a global universal family

$$f : X \to N.$$

The moduli space N of K3 surfaces with a fixed isomorphism of lattices $\varphi : \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda$ exists as a 20-dimensional not Hausdorff complex manifold. (When gluing together open sets, it sometimes happens that what emerges is not Hausdorff, e.g. a line with a double point.)

One can define a universal identification of $\mathrm{H}^2(X, \mathbb{Z})$ with Λ by gluing isomorphisms of the local systems $\mathrm{R}^2 f_* \underline{\mathbb{Z}} \xrightarrow{\sim} \underline{\Lambda}$. So one can define a global period map. For more details on this construction, see Chapter 6, Section 3.3 in Huybrechts' notes [4] and also references there.

Theorem 2.3.6 (Surjectivity of the period map)**.** The global period map

$$\mathcal{P} : N \to D$$

is surjective and a local isomorphism.

Remark 2.3.1. Many topological invariants being equal, one could wonder whether it is possible to distinguish K3 surfaces somehow on the topological level. The answer is no. In fact, all complex K3 surfaces turn out to be deformation equivalent, hence diffeomorphic. The proof of this fact can be divided into three steps:

1. Any K3 surface is deformation equivalent to a K3 surface with the Picard group generated by a line bundle of square 4. (This part relies on the Local Torelli Theorem 2.3.4.)

2. Any K3 surface with the Picard group of the above form is a quartic surface in $\mathbb{P}^3$.

3. Any two smooth quartics in $\mathbb{P}^3$ are deformation equivalent.

Polarized K3 surfaces. One would like to have injective period maps. It is possible if we endow a K3 surface with additional structure, namely polarization.

Definition 2.3.4. A *polarized* K3 surface is a K3 surface X with a fixed ample line bundle $\mathcal{L}$. The K3 surface is *primitively polarized* if $\mathcal{L}$ is primitive, i.e. it is not a tensor power of another ample line bundle.

Note that not all K3 surfaces are polarizable.

Let $(X, \mathcal{L})$ be a primitively polarized K3 surface such that $\mathcal{L}.\mathcal{L} = 2d$. Fix an isomorphism $\Lambda \cong \mathrm{H}^2(X, \mathbb{Z})$, so that Λ becomes a Hodge structure. Recall that by Lefschetz theorem on $(1,1)$-classes, $\mathrm{Pic}(X) \cong \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X)$, so we can associate to the line bundle $\mathcal{L}$ a vector v in the space of Hodge classes of Λ. Then $v^\perp$ is a sublattice of Λ, denote it by $\Lambda_d \stackrel{\text{def}}{=} v^\perp$. Note that $\mathbb{P}(\Lambda_d)$ is a hypersurface in $\mathbb{P}(\Lambda)$, i.e. it is a 20-dimensional complex projective space. The corresponding period domain will be $D_d \stackrel{\text{def}}{=} D \cap \mathbb{P}(\Lambda_d)$, it is now a 19-dimensional complex manifold.

Similarly, one can construct the moduli space

$$N_d \stackrel{\text{def}}{=} \left\{ (X, \mathcal{L}, \varphi) \;\middle|\; \begin{array}{l} X \text{ — a K3 surface,} \\ \mathcal{L} \text{ — a primitive ample line bundle on } X, \\ \varphi : \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda \text{ — an isometry of lattices,} \\ \varphi(\mathcal{L}) = v \end{array} \right\}$$

of primitively polarized K3 surfaces, where v is a fixed class of square $2d$. This moduli space N_d is a fine moduli space, and it turns out to be a Hausdorff complex manifold.

I can define a certain group action on this moduli space. Namely, let O be the group of orthogonal transformations of Λ_d which come from orthogonal transformations of Λ fixing v. Then an element $g \in \mathrm{O}$ acts on a point of N_d as follows:

$$g \cdot (X, \mathcal{L}, \varphi) = (X, \mathcal{L}, g \circ \varphi).$$

Also, D_d obviously carries a O-action, and the period map is O-equivariant, so we can obtain two period maps at once:

$$\begin{aligned} &\mathcal{P}_d : N_d \to D_d, \\ &\bar{\mathcal{P}}_d : \mathrm{O} \setminus N_d \to \mathrm{O} \setminus D_d. \end{aligned}$$

By the Baily–Borel theorem (a reference for which can be [1]), $\mathrm{O} \setminus D_d$ is a normal quasiprojective variety.

One can say even more about the latter period maps, see the forthcoming theorem.

Theorem 2.3.7 (Global Torelli theorem)**.** The period maps

$$\begin{aligned} &\mathcal{P}_d : N_d \to D_d, \\ &\bar{\mathcal{P}}_d : \mathrm{O} \setminus N_d \to \mathrm{O} \setminus D_d. \end{aligned}$$

are injective.

One immediately has a corollary of this.

Corollary 2.3.8 (Global Torelli theorem)**.** Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be two polarized complex K3 surfaces. Then $(X, \mathcal{L}) \cong (X', \mathcal{L}')$ if and only if there exists a Hodge isometry $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(X', \mathbb{Z})$ mapping l to l', where l and l' are cohomology classes of $\mathcal{L}$ and $\mathcal{L}$, respectively.

As we have mentioned above, not all of K3 surfaces admit a polarization, and the period maps are not surjective. But relaxing the condition of being polarizable to that of being quasipolarizable, one can argue that all K3 surfaces are quasipolarizable and the new period maps are isomorphisms.

References

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