

QUOT SCHEMES

SIDDHARTH VENKATESH

ABSTRACT. These are notes for a talk on Quot schemes given in the Spring 2016 MIT-NEU graduate seminar on the moduli of sheaves on $K3$ -surfaces. In this talk, I present the construction of Quot schemes given in [Nit]. I give a definition of the moduli functor represented by the Quot scheme and show that this functor is a closed subfunctor of a relative Grassmanian, whose construction I describe as well.

CONTENTS

1. Introduction	1
2. Representable Functors	2
3. Grassmanians	3
Construction of the Grassmanian over \mathbb{Z}	3
Some Properties of the Grassmanian	3
The Universal Quotient	4
Projectivity	4
4. The Quot Functor and the Hilbert Polynomial Stratification	5
5. The Representability Theorem	6
Embedding into Grassmanian via Castelnuovo-Mumford Regularity	7
Locally Closedness via Flattening Stratification	9
Valuative Criterion for Properness	10
6. Castelnuovo-Mumford Regularity	10
References	14

1. Introduction

Throughout this talk S will be a Noetherian scheme, X will be a projective scheme of finite type over S , \mathcal{E} will be a coherent sheaf on X and L will be a relatively very ample line bundle on X , unless specified otherwise. Also, vector bundle will be used to mean locally free sheaf in this talk. Finally, if T is a scheme over S and \mathcal{F} is a sheaf on S , then \mathcal{F}_T will denote the pullback of \mathcal{F} .

The Hilbert scheme of X over S parametrizes closed subschemes of X . Alternatively, you can view the Hilbert scheme as parameterizing (flat) quotients of the structure sheaf of X . The Quot scheme is a generalization of the Hilbert scheme, where we replace \mathcal{O}_X by \mathcal{E} , a coherent sheaf on X . The goal of this talk is to show that the Quot “scheme”, which we will define via a moduli functor, is indeed representable by a projective scheme over S . To do so, I will show that the Quot functor is a closed subfunctor of a relative Grassmanian over S .

These notes are organized as follows. First, I will define the notion of a representable functor. Subsequently, I will recall the construction of the Grassmanian and describe the moduli functor that it represents by constructing the universal quotient over the Grassmanian. Following this, I will define the moduli functor that the Quot scheme is supposed to represent and use the notion of Hilbert polynomial to get a stratification on the Quot scheme. As an example, I will also show that the Grassmanian is an example of such a subfunctor.

Next, I will state a corollary of Castelnuovo-Mumford regularity and use it to embed the Quot functor as a subfunctor of a relative Grassmanian over S . Using, without proof, the construction of the

flattening stratification, I will then show that Quot is a locally closed subfunctor of the Grassmanian, which will finish the proof of representability. Projectivity will follow from showing that the Quot satisfies the valuative criterion for properness. Finally, I will end the talk with a discussion of Castelnuovo-Mumford regularity.

2. Representable Functors

Let F be a contravariant functor from the category of schemes over S to sets.

Definition 2.1. We say that F is representable if there exists some scheme Y/S such that F is isomorphic to $\text{Hom}_S(-, Y)$.

Note that by the Yoneda lemma, the scheme Y can be recovered up to unique isomorphism from the functor F .

Example 2.2. Here are some well known examples of representable functors.

1. Projective space: The functor represented by \mathbb{P}_S^n sends T/S to the set

$$\{(\mathcal{F}, l_0, \dots, l_n) : \mathcal{F} \text{ is an invertible sheaf on } T, l_i \text{ are generating sections of } \mathcal{F}\}$$

up to isomorphisms of invertible sheaves. Morphisms go to pullbacks of invertible sheaves. The data of $n + 1$ generating global sections of \mathcal{F} is the same as the data of a surjection

$$\oplus^{n+1} \mathcal{O}_T \rightarrow \mathcal{F}.$$

However, the set $\text{Hom}(T, \mathbb{P}^n)$ shouldn't just consist of all such surjections because if we replace \mathcal{F} by an isomorphic line bundle \mathcal{F}' and the sections l_i by the corresponding sections on \mathcal{F}' , then the morphism to \mathbb{P}^n is unchanged. This is the same as having a commutative diagram

$$\begin{array}{ccc} \oplus^{n+1} \mathcal{O}_T & \xrightarrow{q} & \mathcal{F} \\ & \searrow q' & \downarrow \sim \\ & & \mathcal{F}' \end{array}$$

But such a diagram exists if and only if $\ker q = \ker q'$. Hence,

$$\text{Hom}(T, \mathbb{P}^n) = \{(\mathcal{F}, q) : \mathcal{F} \text{ an invertible sheaf on } T, q : V \rightarrow \mathcal{F} \text{ an epimorphism}\} / \sim$$

where the equivalence relation is equality of kernels.

2. Relative projective space: Let V be a vector bundle (locally free sheaf) on S . Then, the functor represented by $\mathbb{P}(V)$ sends T/S to the set

$$\{(\mathcal{F}, q) : \mathcal{F} \text{ an invertible sheaf on } T, q : V \rightarrow \mathcal{F} \text{ an epimorphism}\} / \sim$$

with $(\mathcal{F}, q) \sim (\mathcal{F}', q')$ if $\ker q = \ker q'$.

3. Grassmanians: We will elaborate on the Grassmanian later. Here, we just state the moduli functor represented by the Grassmanian. The grassmanian functor $\text{Grass}_S(n, r)$ sends T/S to the set

$$\{(\mathcal{F}, q) : \mathcal{F} \text{ a locally free sheaf of rank } r \text{ on } T, q : \mathcal{O}_T^n \rightarrow \mathcal{F} \text{ an epimorphism}\} / \sim$$

with the same equivalence relation as before. Again, we can also define relative Grassmanians. Given a locally free sheaf V on S and an integer r , we can define $\text{Grass}(V, r)$ by replacing \mathcal{O}_T^n with V_T in the above definition.

Since the relative Grassmanian is of critical importance to the proof of representability of the Quot functor, we give a detailed construction of the projective scheme that represents it. However, in the following section, we merely give a construction of the Grassmanian for V a trivial vector bundle. The general case can be constructed either in a completely analogous manner or by using the fact that the relative Grassmanian is locally the same as the ordinary Grassmanian and then using its universal property to patch the local Grassmanians together.

3. Grassmanians

Construction of the Grassmanian over \mathbb{Z} . The construction of the Grassmanian is motivated by the identification of the Grassmanian as the quotient $M_{r \times n}/GL_r$ but we give an elementary description of the construction via gluing together of affine patches.

Let I be a subset of $\{1, \dots, n\}$ of size r . For any $r \times n$ matrix M , let M_I denote the $r \times r$ minor of M consisting of the columns labeled by I . Let X^I be the $r \times n$ matrix defined by setting X_I^I as the identity and letting the remaining entries be independent variables $x_{p,q}^I$ over \mathbb{Z} . Let $\mathbb{Z}[X^I]$ be the polynomial ring in the x^I and let $U^I = \text{Spec} \mathbb{Z}[X^I]$. (This is supposed to be the affine subspace of the Grassmanian represented by matrices whose I th minor is invertible.)

The Grassmanian is constructed by gluing together the U^I . Let J now be any size r subset of $\{1, \dots, n\}$. Let P_J^I be the polynomial in the variables x^I obtained by taking the determinant of X_J^I and let U_J^I be the open subset of U^I where P_J^I is nonzero. We define the gluing map

$$\Theta_{I,J} : U_J^I \rightarrow U_I^J$$

via the ring homomorphism

$$\theta_{I,J} : \mathbb{Z}[X_I^J, 1/P_I^J] \rightarrow \mathbb{Z}[X_J^I, 1/P_J^I]$$

where the images of $x_{p,q}^J$ are given by the entries of the matrix

$$(X_J^I)^{-1} X^I.$$

This is well-defined because the only polynomials appearing in the denominators in $(X_J^I)^{-1}$ are P_J^I (think Cramer's rule), and because this map sends

$$P_I^J = \det(X_I^J) \mapsto \det(X_J^I)^{-1} \det(X_I^I) = 1/P_J^I.$$

Exercise Show that the maps $\Theta_{I,J}$ satisfy the cocycle condition.

The idea is that at the level of points $\Theta_{I,J}$ sends an element of the Grassmanian represented by a matrix whose I th minor is the identity to the unique matrix whose J th minor is the identity and which also represents the same element of the Grassmanian.

Hence the affine patches glue together to give a scheme over S which we will denote as the Grassmanian $\text{Grass}(n, r)$.

Some Properties of the Grassmanian. The above construction of the Grassmanian allows us to immediately see some of its properties, which we state here without proof.

1. Grass is smooth of relative dimension $r(n - r)$.
2. Grass is separated: This follows because the intersection of the diagonal with $U^I \times U^J$ is the vanishing locus defined by the entries of the matrix formula

$$X_I^J X^I - X^J = 0.$$

3. Grass is proper: This can be checked using the valuative criterion for properness.

The Universal Quotient. To show that the Grassmanian we constructed represents the functor we mentioned earlier, we construct a universal rank r vector bundle \mathcal{U} over the Grassmanian equipped with a canonical map $\mathcal{O}_{\text{Grass}}^n \rightarrow \mathcal{U}$, and the correspondence with the aforementioned functor sends morphisms to pullbacks of this data. This universal vector bundle and quotient map will be constructed using the same affine patches we used to construct $\text{Grass}(n, r)$. The main idea is that on each patch, elements of the Grassmanian are naturally represented by a unique matrix.

Let $\mathcal{U}|_{U^I}$ be $\oplus^r \mathcal{O}_{U^I}$ and define the map

$$q|_{U^I} : \oplus^n \mathcal{O}_{U^I} \rightarrow \oplus^r \mathcal{O}_{U^I}$$

via the matrix X^I .

Define the transition isomorphisms

$$\rho_{I,J} : \oplus^r \mathcal{O}_{U^I} = \mathcal{U}|_{U^I} \rightarrow \mathcal{U}|_{U^J} = \oplus^r \mathcal{O}_{U^J}$$

via the matrix $(X_J^I)^{-1} \in GL_d(U_J^I)$.

Exercise: Check that the $\rho_{I,J}$ satisfy the cocycle condition and hence the $\mathcal{U}|_{U^I}$ glue together to give a rank r vector bundle \mathcal{U} on Grass . Additionally, check that $q|_{U^I}$ glue together to give a morphism $\oplus^n \mathcal{O}_{\text{Grass}} \rightarrow \mathcal{U}$.

We can now see that Grass represents the functor described in the previous section. Let Y be a scheme over S and let $f : Y \rightarrow \text{Grass}(n, r)$ be a morphism. Then, we get a locally free sheaf of rank r on Y by taking $f^*\mathcal{U}$ and, since f^* is a right-exact functor, we get an epimorphism $f^*q : \mathcal{O}_Y^n \rightarrow f^*\mathcal{U}$. This gives us a map from $\text{Hom}(-, \text{Grass}) \rightarrow \text{Grass}$ (where the second Grass is the functor described before). We now describe a map in the reverse direction.

Suppose we have a surjection $q : \mathcal{O}_Y^n \rightarrow \mathcal{F}$ with \mathcal{F} of rank r . This is given by choosing n (ordered) global sections of \mathcal{F} which generate \mathcal{F} at each point. Let Y^I be the open subset of Y where the I global sections of \mathcal{F} generate. Then, over Y^I , $\mathcal{F} \cong \mathcal{O}_{Y^I}^r$. Thus, over Y^I , the map q is given by an $r \times n$ matrix M^I valued in $H^0(Y^I, \mathcal{O}_Y)$.

We define the map $Y^I \rightarrow U^I$ by sending $x_{p,q}^I$ to the entries of the matrix M^I . This map glues together to give a map from Y to $\text{Grass}(n, r)$ because the transition matrix in $Y^I \cap Y^J$ is precisely $(M_J^I)^{-1}$ (as we are switching from the basis of the I -th global sections to the basis of the J -th global sections.) This gives us a map in the reverse direction and it is easy to see that the two maps are mutually inverse.

Projectivity. We end our discussion of the Grassmanian by describing the Plucker embedding. $\text{Grass}(n, r)$ embeds into $\mathbb{P}(\det \mathcal{U})$, where $\det \mathcal{U}$ is the determinantal line bundle of \mathcal{U} whose transition functions are given by $\det \rho_{I,J} = 1/P_J^I$. For each I define a global section $\sigma^I \in H^0(\text{Grass}, \det \mathcal{U})$ as

$$\sigma^I|_{U^J} = P_I^J.$$

Exercise: Check that each $\sigma^I|_{U^J}$ glues together to give a global section σ^I . Check that the collection of all σ^I defines a linear system which is base point free and separates points relative to $\text{Spec} \mathbb{Z}$.

Thus, we have an embedding, necessarily closed, of $\text{Grass}(n, r)$ into $\mathbb{P}^{\binom{n}{r}-1}$. In particular, $\det \mathcal{U}$ is a relatively very ample line bundle on $\text{Grass}(n, r)$ over \mathbb{Z} .

Remark. All the above constructions work if we replace $\mathcal{O}_{\mathbb{Z}}^n$ with any rank n vector bundle V . We will get a $\text{Grass}(V, r)$ with a universal quotient \mathcal{U} which satisfies the universal property mentioned in the previous section. Additionally, $\text{Grass}(V, r)$ will embed into $\mathbb{P}(\pi_* \mathcal{U})$, where π is the projection $\text{Grass}(V, r) \rightarrow \mathbb{Z}$.

Additionally, we can go further and replace the vector bundle V with any coherent sheaf \mathcal{E} over $\text{Spec} \mathbb{Z}$. In this case, the functor represented by $\text{Grass}(\mathcal{E}, r)_S$ sends T/S to the set

$$\{(\mathcal{F}, q) : \mathcal{F} \text{ a locally free sheaf of rank } r \text{ over } T, q : \mathcal{E}_T \rightarrow \mathcal{F} \text{ an epimorphism}\}$$

up to the same equivalence relation as before. But we will not need this and hence will not go into the details.

4. The Quot Functor and the Hilbert Polynomial Stratification

In a way, the Quot functor is a generalization of the Grassmanian. Recall that S is a Noetherian scheme, X is a projective scheme over S and \mathcal{E} is a locally free sheaf over X . We first begin with a notion of flatness.

Definition 4.1. If $f : X \rightarrow S$ is a scheme of finite type over S and \mathcal{F} is a quasi-coherent sheaf on X , then we say that \mathcal{F} is flat over S if for each $x \in X$, \mathcal{F}_x is flat over $S_{f(x)}$.

We now define the moduli functor represented by the Quot scheme.

Definition 4.2. The functor $\text{Quot}_{\mathcal{E}/X/S}$ sends T/S to the set of all equivalence classes $\langle \mathcal{F}, q \rangle$, where:

1. \mathcal{F} is a coherent sheaf on X_T flat over T and with proper scheme theoretic support over T .
2. q is an epimorphism $\mathcal{E}_{X_T} \rightarrow \mathcal{F}$.
3. The equivalence relation is equality of kernels $(\mathcal{F}, q) \sim (\mathcal{F}', q')$ if $\ker q = \ker q'$. Equivalently, $(\mathcal{F}, q) \sim (\mathcal{F}', q')$ if there exists an isomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ compatible with q and q' .

If $E = \mathcal{O}_X$, then $\text{Quot}_{\mathcal{O}_X/X/S}$ is also denoted by $\text{Hilb}_{X/S}$. In this case, $\text{Hilb}_{X/S}$ can be seen as the moduli space parameterizing all closed subschemes of X that are flat and proper over S .

Before we give some examples, we need to describe a decomposition of the Quot scheme into a (possibly infinite) disjoint union of subschemes that comes from the notion of Hilbert polynomial. Let X be a finite type scheme over a field k and let L be a line bundle over X . If \mathcal{F} is a coherent sheaf over X , let $\mathcal{F}(m)$ denote $\mathcal{F} \otimes L^{\otimes m}$.

Definition 4.3. If the support of \mathcal{F} is proper over k , then define the Hilbert polynomial $\Phi \in \mathbb{Q}[t]$ of \mathcal{F} as

$$\Phi(m) = \chi(\mathcal{F}(m)) = \sum_{i=0}^{\dim \mathcal{F}} (-1)^i h^i(X, \mathcal{F}(m)).$$

The fact that $\Phi(m)$ is polynomial in m we leave unproven. If L is a very ample line bundle, then this notion of Hilbert polynomial coincides with ordinary notion of the Hilbert polynomial of a sheaf on a projective scheme.

Now, if X is a finite type scheme over S , L is a line bundle over X and \mathcal{F} is a coherent sheaf with proper support over S , then, applying the definition above to $\mathcal{F}|_{X_s}$ and $L|_{X_s}$ for $s \in S$, we get a family of Hilbert polynomials parameterized by the points of S , which we denote as Φ_s .

Key Fact: If \mathcal{F} is flat over S , then Φ_s is locally constant on S .

Thus, we see that for connected T/S , every quotient sheaf in $\text{Quot}_{\mathcal{E}/X/S}(T)$ has a unique Hilbert polynomial (it depends on the particular quotient sheaf but is the same above all points in T). Hence, we see that for a fixed L ,

$$\text{Quot}_{\mathcal{E}/X/S} = \coprod_{\Phi \in \mathbb{Q}[t]} \text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$$

where $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is the functor that only allows \mathcal{F} with Hilbert polynomial Φ . A few remarks on this decomposition are in order.

Remark. 1. Note that the coproduct of contravariant functors from schemes to sets is given by taking disjoint unions in the image when evaluating at connected schemes. If we have disconnected schemes, then we take the disjoint union in the image on each connected

component and then take the product. This is exactly what is going on above in the decomposition of Quot.

2. The decomposition depends in a fundamental manner on the line bundle L that we choose. For example, if we choose $L = \mathcal{O}_X$, then the only polynomials for which the Quot is nonempty are the constant ones. In particular,

$$\text{Quot}_{\mathcal{E}/X/S}^{m, \mathcal{O}_X}$$

consists of the quotients whose global sections along each fiber are of rank m . If $X = S$, then global sections along fiber correspond just to the fiber of the sheaves and hence, $\text{Quot}_{\mathcal{E}/S/S}^{m, \mathcal{O}_X}$ parameterizes locally free quotients of E of rank m .

3. The decomposition above need not be into connected components.

Here is an example.

Example 4.4. The functor $\text{Quot}_{\mathcal{O}_S^n/S/S}^r$ sends T to the equivalence classes $\langle \mathcal{F}, q \rangle$ where \mathcal{F} is a locally free sheaf on T of rank r and q is an epimorphism $\mathcal{O}_T^n \rightarrow \mathcal{F}$ (up to the standard equivalence relation). Thus, we see that this Quot functor is represented by $\text{Grass}(n, r)$. More generally, replacing \mathcal{O}_S^n with some vector bundle E on S gives us $\text{Grass}(V, r)$.

Our goal is to show that for any choice of L, Φ , $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is representable by a closed subscheme of the Grassmanian of some vector bundle over S . This will immediately imply representability of $\text{Quot}_{\mathcal{E}/X/S}$ as a whole. To do so, we will work with the moduli functor represented by the Grassmanian, rather than the geometric space directly. Here is an outline of the proof of representability:

1. We will begin by embedding $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ as a subfunctor of a Grassmanian functor. To do so, we will need a uniform vanishing result on cohomology which we will obtain from a technical result known as Castelnuovo-Mumford regularity (we will go into this in detail).
2. We then show that $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is a locally closed subfunctor of Grass (a formal definition of what this means will be given in the relevant section). This will prove that the Quot scheme is representable by a locally closed subscheme of the Grassmanian. For this result, we will use, without proof, an important technical result known as the flattening stratification (the proof is too technical for this talk).
3. Finally, to show that Quot is a closed subscheme, we will use the valuative criterion for properness.

5. The Representability Theorem

In this talk, the representability theorem we will prove is due to Altman and Kleiman.

Theorem 5.1. Let S be a Noetherian scheme, X a closed subscheme of $\mathbb{P}(V)$ for some vector bundle V on S , $L = \mathcal{O}_{\mathbb{P}(V)}(1)|_X$, \mathcal{E} a coherent quotient sheaf of $\pi^*(W)(n)$ (π being the projection $X \rightarrow S$) where W is some vector bundle on S and n is an integer. Let $\Phi \in \mathbb{Q}[t]$. Then, the functor $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is representable by a closed subscheme of $\text{Grass}(V', n')$ where V' is a vector bundle over S that is an exterior power of the tensor product of W with symmetric powers of V .

This theorem is a little less general than that of Grothendieck's, in which \mathcal{E} can be taken to be any coherent sheaf and V can be replaced by any coherent sheaf and the Grassmanian is now the Grassmanian of a coherent sheaf and not necessarily a vector bundle, but the more general theorem can be proved in a similar manner (see [Nit, pp. 27-28]) and we will not go into the details.

Let us begin the proof of the theorem by reducing to the case of $\text{Quot}_{\pi^*W/\mathbb{P}(V)/S}^{\Phi,L}$. We first need the notion of (locally) closed subfunctors.

Definition 5.2. Let G be a contravariant functor from schemes over S to sets and let F be a subfunctor (i.e. there exists a natural transformation $F \rightarrow G$ that is injective when evaluated at any scheme T/S). We say that F is a (locally) closed subfunctor of G if for every T/S , there exists a (locally) closed subscheme T'/S such that the functor $F \times_G \text{Hom}_S(-, T)$ is represented by T' .

The key fact is that if G is represented by some scheme Y/S and F is a (locally) closed subfunctor of G , then F is represented by a (locally) closed subscheme of Y . To find this subscheme, just apply the above definition to Y in the place of T . We use this to do the necessary reduction to $\text{Quot}_{\pi^*W/\mathbb{P}(V)/S}^{\Phi,L}$.

Lemma 5.3. (a) Tensoring by L^{-n} gives an isomorphism $\text{Quot}_{\mathcal{E}/X/S}^{\Phi,L} \rightarrow \text{Quot}_{\mathcal{E}(-n)/X/S}^{\Psi,L}$ where $\Psi(m) = \Phi(m - n)$.

(b) Let $\psi : \mathcal{E} \rightarrow \mathcal{G}$ be an epimorphism of coherent sheaves on X . Then, the corresponding natural transformation

$$\text{Quot}_{\mathcal{G}/X/S}^{\Phi,L} \rightarrow \text{Quot}_{\mathcal{E}/X/S}^{\Phi,L}$$

is a closed embedding.

Proof. (a) is obvious. We just prove (b). For this we need the construction of the vanishing scheme $V(\phi)$ of a morphism of coherent sheaves $\phi : \mathcal{F} \rightarrow \mathcal{F}'$ on an arbitrary scheme Y/S . This scheme will have the universal property a morphism $f : Z \rightarrow Y$ factors through $V(\phi)$ if and only if $f^*\phi = 0$. The construction is a little involved so we leave the proofs to [Nit, pp. 16-17] and simply state the result we need here.

Proposition 5.4. Let S be a Noetherian scheme and $\pi : X \rightarrow S$ a projective morphism. Let \mathcal{F} and \mathcal{F}' be coherent sheaves on X with \mathcal{F}' flat over S . Then, the contravariant functor $\text{hom}(\mathcal{F}, \mathcal{F}')$ which assigns to each T/S the set $\text{Hom}(\mathcal{F}_{X_T}, \mathcal{F}'_{X_T})$ is represented by some $Z = \text{Spec}(\text{Sym } \mathcal{Q})$ where \mathcal{Q} is a coherent sheaf on S . In addition, the closed subscheme Z_0 defined by the vanishing of the ideal sheaf generated by \mathcal{Q} is the closed subscheme where the universal homomorphism vanishes (which represents the identity on Z).

As a result of this proposition, the vanishing scheme of a particular morphism $\phi : \mathcal{F} \rightarrow \mathcal{F}'$ is given by the preimage $g^{-1}(Z_0)$ where $g : Y \rightarrow Z$ is the morphism determined by ϕ .

Getting back to the proof of (b), we note that what we need is the following result: given any scheme T over S and given $\langle \mathcal{F}, q \rangle \in \text{Quot}_{\mathcal{E}/X/S}^{\Phi,L}(T)$ defining a morphism $\text{Hom}_S(-, T) \rightarrow \text{Quot}_{\mathcal{E}/X/S}^{\Phi,L}$ by the Yoneda lemma, there exists a closed subscheme T' of T that represents

$$\text{Quot}_{\mathcal{G}/X/S}^{\Phi,L} \times_{\text{Quot}_{\mathcal{E}/X/S}^{\Phi,L}} \text{Hom}_S(-, T).$$

Let us see what this functor gives us when evaluated at Y/S . We get the set

$$\{(\langle \mathcal{F}', q' \rangle, f) : f \in \text{Hom}_S(Y, T), \mathcal{F}' = f^*\mathcal{F}, q' = f^*q \text{ s.t. } \ker q' \supseteq \ker \psi\}.$$

Thus, we need to show that there exists a closed subscheme T' of T such that a morphism $f : Y \rightarrow T$ factors through T' if and only if f^*q is 0 on $\ker \psi$ i.e. if and only if $f^*q|_{\ker \psi} = 0$. Thus, T' does in fact exist as it is the vanishing scheme of $q|_{\ker \psi}$. □

Embedding into Grassmanian via Castelnuovo-Mumford Regularity. Hence, to prove Theorem 5.1, we can now assume $X = \mathbb{P}(V)$ and $\mathcal{E} = \pi^*W$. The next step is to embed $\text{Quot}_{\mathcal{E}/X/S}^{\Phi,L}$ into the Grassmanian of an exterior power of the tensor product of W and a symmetric power of V . To do so, we will use the following result which we later prove using Castelnuovo-Mumford regularity.

Proposition 5.5. There exists some integer m that depends only on the rank of V , the rank of W and Φ such that for all $r \geq m$, for all schemes T/S and for all T -flat coherent quotients \mathcal{F} of \mathcal{E}_{X_T} with kernel \mathcal{G} , the following facts hold:

1. $\pi_{T*}(\mathcal{F}(r)), \pi_{T*}(\mathcal{G}(r)), \pi_{T*}(\mathcal{E}_{X_T}(r))$ are locally free sheaves of ranks determined by the rank of V , rank of W , r and Φ , with $\pi_T^*(\mathcal{F}(r))$ of rank $\Phi(r)$ in particular. In addition, all higher direct image sheaves vanish.
2. In the following commutative diagram of locally free sheaves on X_T ,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_T^* \pi_{T*}(\mathcal{G}(r)) & \longrightarrow & \pi_T^* \pi_{T*}(\mathcal{E}_{X_T}(r)) & \longrightarrow & \pi_T^* \pi_{T*}(\mathcal{F}(r)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}(r) & \longrightarrow & \mathcal{E}_{X_T}(r) & \longrightarrow & \mathcal{F}(r) & \longrightarrow & 0
\end{array}$$

the rows are exact and the vertical maps are surjective.

The proof of this proposition uses a flat base change result applied to \mathcal{F} and some cohomological vanishing results which will be given to us by Castelnuovo-Mumford regularity. Since this result is a little technical, it is left to the end of the talk. For now, we use this proposition to embed the Quot functor into the Grassmanian.

This embedding is done as follows. Fix some positive integer $r \geq m$ and consider some scheme T/S . Suppose we have some $\langle \mathcal{F}, q \rangle$ in $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}(T)$. Then, twisting by L^r and applying π_{T*} gives us an epimorphism

$$\pi_{T*}(q(r)) : \pi_{T*}(\mathcal{E}_{X_T}(r)) \rightarrow \pi_{T*}(\mathcal{F}(r))$$

because all the higher direct images vanish. If $\langle \mathcal{F}', q' \rangle$ is some other element of $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}(T)$ with $\ker(q) = \ker(q')$, then we will also have

$$\ker \pi_{T*} q = \pi_{T*} \ker q = \pi_{T*} \ker q' = \ker \pi_{T*} q'.$$

Hence, π_* gives us a natural transformation

$$\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L} \rightarrow \text{Grass}(\pi_*(\pi^*(W) \otimes L^r), \Phi(r)) = \text{Grass}(W \otimes \text{Sym}^r V, \Phi(r)).$$

We can show that this natural transformation is injective. To do so, we need to be able to recover q from $\pi_{T*}(q(r))$. It suffices to recover $q(r)$ from $\pi_{T*}(q(r))$. Note that $q(r)$ is the cokernel of the map from $\mathcal{G}(r) \rightarrow \mathcal{E}_{X_T}(r)$. Since the vertical maps in the commutative diagram above are surjective, this is the cokernel of the composite map

$$\pi_T^* \pi_{T*}(\mathcal{G}(r)) \rightarrow \mathcal{G}(r) \rightarrow \mathcal{E}_{X_T}(r)$$

which is the same as the composite map

$$\pi_T^* \pi_{T*}(\mathcal{G}(r)) \rightarrow \pi_T^* \pi_{T*}(\mathcal{E}_{X_T}(r)) \rightarrow \mathcal{E}_{X_T}(r).$$

Thus, to recover q from $\pi_{T*}(q(r))$ it suffices to recover the first map in the above composition. This follows from the fact that the first map is π_T^* of the inclusion of the kernel of $\pi_{T*}(q(r))$. Hence, we see that we can embed

$$\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L} \rightarrow \text{Grass}(W \otimes \text{Sym}^r V, \Phi(r))$$

for some r sufficiently large.

Locally Closedness via Flattening Stratification. Our next goal is to show that this embedding actually presents $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ as a locally closed subfunctor of $\text{Grass}(W \otimes \text{Sym}^r V, \Phi(r))$. This will require a technical result known as the flattening stratification which we state now without proof. A reference for the local case can be found in [Nit, Theorem 4.3]. Gluing issues are handled by the universal property in the theorem.

Proposition 5.6. Let S be a Noetherian scheme and let \mathcal{F} be a coherent sheaf on X/S , where X is a closed subscheme of $\mathbb{P}(V)$ for some vector bundle V on S . Then, the set I of Hilbert polynomials of restrictions of \mathcal{F} to the fibers of the map from X to S is finite. Moreover, for each $\Phi \in I$, there exists a locally closed subscheme S_Φ of S such that the following three properties hold:

1. **Point Set:** The underlying set $|S_\Phi|$ consists of all points $s \in S$ above which the Hilbert polynomial of \mathcal{F} is Φ .
2. **Universal Property:** Let S' be the scheme theoretic disjoint union of the S_Φ over all Φ in I and let $f : S' \rightarrow S$ be the natural surjection. Then, the sheaf $f^*\mathcal{F}$ on $\mathbb{P}(f^*V)$ is flat on S' . Moreover $f : S' \rightarrow S$ has the universal property that a morphism $g : T \rightarrow S$ factors through f if and only if $g^*\mathcal{F}$ is flat over T (we use g to denote the base changed morphism on the projective space as well). The subscheme S_Φ is thus uniquely determined by Φ .
3. **Closure of Strata:** Let the set I be given the total order by putting $\Phi < \Phi'$ if this holds for sufficiently large $n \in \mathbb{Z}$. Then, the closure of S_Φ consists of the union of all $S_{\Phi'}$ for $\Phi' \geq \Phi$.

Let us now use this proposition to show that $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is a locally closed subfunctor of $\text{Grass}(W \otimes \text{Sym}^r V, \Phi(r))$. Again, if we look at the definition, what we need to show is that given T/S and some $(\mathcal{F}, q) \in \text{Grass}(W \otimes \text{Sym}^r V, \Phi(r))(T)$, there exists a locally closed subscheme T' of T representing

$$\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L} \times_{\text{Grass}(\pi_*(\mathcal{E}(r)), \Phi(r))} \text{Hom}_S(-, T).$$

We define T' as follows. Define \mathcal{G} as the kernel of

$$q : \pi_{T*}(\mathcal{E}_{X_T}(r)) = \pi_*(\mathcal{E}(r))_T \rightarrow \mathcal{F}$$

and let h be the composite map

$$\pi_T^*\mathcal{G} \rightarrow \pi_T^*\pi_{T*}(\mathcal{E}_{X_T}(r)) \rightarrow \mathcal{E}_{X_T}(r).$$

Let \mathcal{J} be the cokernel of h and define T' to be the subscheme obtained in the flattening stratification for \mathcal{J} corresponding to the polynomial Φ . We claim that T' represents the above functor.

Let Y be a scheme over S . By the universal property of the flattening stratification, a morphism $f : Y \rightarrow T$ factors through T' if and only if $f^*\mathcal{J}$ is flat with Hilbert polynomial Φ (in the second half, f denotes that base changed morphism between $X_Y \rightarrow X_T$). Let us reframe this latter property. First, we note that

$$f^*\mathcal{J} = f^*(\text{coker } h) = \text{coker}(f^*h).$$

Writing h out, we see that

$$f^*\mathcal{J} = \text{coker}(f^*\pi_T^* \ker q \rightarrow f^*\pi_T^*\pi_{T*}(\mathcal{E}_{X_T}) \rightarrow f^*\mathcal{E}_{X_T}).$$

Because π_{T*} and π_{Y*} have no higher direct images on the relevant sheaves by the Castelnuovo-Mumford regularity result stated before,

$$f^*\pi_T^* = \pi_Y^*f^*$$

and

$$f^*\pi_{T*} = \pi_{Y*}f^*.$$

Hence, we see that

$$f^*\mathcal{J} = \text{coker}(\pi_Y^*f^* \ker q \rightarrow \pi_Y^*\pi_{Y*}f^*\mathcal{E}_{X_T} \rightarrow f^*\mathcal{E}_{X_T}).$$

On the other hand, f lives in the image of the above functor if and only if f^*q comes from some $(\mathcal{F}', q') \in \text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}(Y)$. By looking at how we recovered Quot from Grass, this is true if and only if

$$\text{coker}(\pi_Y^* \ker(f^*q) \rightarrow \pi_Y^* f^* \pi_{T*}(\mathcal{E}_{X_T}(r)) \rightarrow f^* \mathcal{E}_{X_T})$$

is flat. Again, using the same identities as before, we can rewrite this as

$$\text{coker}(\pi_Y^* \ker(f^*q) \rightarrow \pi_Y^* \pi_{Y*} f^*(\mathcal{E}_{X_T}(r)) \rightarrow f^* \mathcal{E}_{X_T}).$$

The only difference between this sheaf and $f^*\mathcal{J}$ is that one has $\ker(f^*q)$ and the other has $f^*\ker q$. While these are not the same sheaf for general f , replacing one by the other does not change the cokernel (by right exactness of f^*). Hence, we see that the sheaf above is $f^*\mathcal{J}$. Hence, a morphism $f : Y \rightarrow T'$ factors through T' if and only if $f^*\mathcal{J}$ is flat and has Hilbert polynomial Φ which is true if and only if

$$(\langle f^*\mathcal{F}, f^*q \rangle, f) \in \text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}(Y) \times_{\text{Grass}(\pi_*(\mathcal{E}(r)), \Phi(r))} \text{Hom}_S(Y, T).$$

Hence, $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is a closed subfunctor of $\text{Grass}(\pi_*\mathcal{E}, \Phi(r))$ and is thus represented by the T' obtained by taking T to be the latter scheme with the morphism given by the universal quotient.

Valuative Criterion for Properness. To finish the proof of Theorem 5.1, we need to show that $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is actually represented by a closed subscheme of the Grassmanian. For this, we just need to show that it satisfies the valuative criterion for properness. Namely, if R is a dvr over S with field of fractions K , we need to show that the restriction map

$$\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}(\text{Spec}R) \rightarrow \text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}(\text{Spec}K)$$

is bijective. So, suppose we have some coherent quotient map

$$q : \mathcal{E}_{X_K} \rightarrow \mathcal{F}.$$

Let j be the map from $\text{Spec}K$ to $\text{Spec}R$ and let $\overline{\mathcal{F}}$ be the image of the composition of

$$\mathcal{E}_{X_R} \rightarrow j_* \mathcal{E}_{X_K}$$

with j_*q . Let $\overline{q} : \mathcal{E}_{X_R} \rightarrow \overline{\mathcal{F}}$ be the epimorphism obtained by looking only at the image. Surjectivity of the map will follow if we show $\overline{\mathcal{F}}$ is flat over R because applying j^* makes the first map in the composition the identity and the second map q . But on a dvr, flatness is the same as being torsion free (all higher tor is trivial by dimension reasons). Now, \mathcal{F} is a torsion free sheaf over K and hence $j_*\mathcal{F}$ is a torsion free sheaf over R . Thus, so is $\overline{\mathcal{F}}$. To prove injectivity, note that the morphism between the functors is given by tensoring \mathcal{F}, q with K . Injectivity of the above map follows from the valuative criterion for separatedness: $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is already known to be separated as locally closed subschemes of separated schemes are separated. Hence, $\text{Quot}_{\mathcal{E}/X/S}^{\Phi, L}$ is proper.

This finishes the proof of the theorem, up to the proof of Proposition 5.5. For this proposition, we will need to use some results that are collectively known as Castelnuovo-Mumford regularity.

6. Castelnuovo-Mumford Regularity

Let k be a field and let \mathcal{F} be a coherent sheaf on \mathbb{P}^n over k . Let m be an integer.

Definition 6.1. We say that \mathcal{F} is m -regular if for all $i \geq 1$,

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0.$$

This is a strange definition but it works quite well in practice as it inducts well on dimension of \mathbb{P}^n . More precisely, we have the following lemma.

Lemma 6.2. Let \mathcal{F} be m -regular on \mathbb{P}^n and let H be a hyperplane that contains no associated point of \mathcal{F} (which always exists if k is infinite). Then, $\mathcal{F}|_H$ is also m -regular.

Proof. Since H has no associated points of \mathcal{F} , the ideal sheaf of H is locally generated by elements that have no zerodivisors on \mathcal{F} . Hence, for each i , we get a short exact sequence

$$0 \rightarrow \mathcal{F}(m-i-1) \rightarrow \mathcal{F}(m-i) \rightarrow \mathcal{F}_H(m-i) \rightarrow 0.$$

Taking cohomology gives us a long exact sequence

$$\cdots \rightarrow H^i(\mathbb{P}^n, \mathcal{F}(m-i)) \rightarrow H^i(\mathbb{P}^n, \mathcal{F}_H(m-i)) \rightarrow H^{i+1}(\mathbb{P}^n, \mathcal{F}(m-i-1)) \rightarrow \cdots$$

which proves the claim. \square

This inductive approach allows us to conclude extremely strong consequences of m -regularity. The following lemma is due to Castelnuovo.

Lemma 6.3. If \mathcal{F} is an m -regular sheaf on \mathbb{P}^n , then the following statements hold:

- (a) The canonical map $H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \rightarrow H^0(\mathbb{P}^n, \mathcal{F}(r+1))$ is surjective for $r \geq m$.
- (b) We have $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for all $i \geq 1$ whenever $r \geq m-i$.
- (c) The sheaf $\mathcal{F}(r)$ is generated by global sections for all $r \geq m$.

Proof. Since these properties can be tested after base changing to a field extension we can assume without loss of generality that k is infinite. Let us induct on the dimension of \mathbb{P}^n . These properties hold trivially if $n = 0$ so the base case holds. Now, assume these statements hold for all $m \leq n$. Choose some hyperplane H that contains no associated point of \mathcal{F} . Then, \mathcal{F}_H is also m -regular and hence all three properties hold for \mathcal{F}_H .

Let us now prove (b). For $r = m-i$, by the definition of m -regularity $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$. We now prove (b) by inducting on $r \geq m-i+1$. Again, look at the same long exact sequence as before. We have an exact sequence

$$H^i(\mathbb{P}^n, \mathcal{F}(r-1)) \rightarrow H^i(\mathbb{P}^n, \mathcal{F}(r)) \rightarrow H^i(\mathbb{P}^{n-1}, \mathcal{F}_H(r))$$

The first term is 0 by the induction hypothesis for all $i \geq 1$. The second term is 0 by the inductive hypothesis. Hence, the induction step follows. This proves (b).

For the proof of (a) and (c), we use (b) already proven and look at the following commutative diagram

$$\begin{array}{ccccc} H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) & \xrightarrow{\sigma} & H^0(H, \mathcal{F}_H) \otimes H^0(H, \mathcal{O}(1)) & & \\ & & \downarrow \mu & & \downarrow \tau \\ H^0(\mathbb{P}^n, \mathcal{F}(r)) & \xrightarrow{\alpha} & H^0(\mathbb{P}^n, \mathcal{F}(r+1)) & \xrightarrow{\nu_{r+1}} & H^0(H, \mathcal{F}_H(r+1)) \end{array}$$

By induction, τ is surjective. By part (b), $H^1(\mathbb{P}^n, \mathcal{F}(r-1)) = 0$ for $r \geq m$ and hence ν_r, ν_{r+1} are surjective. Since σ is $\nu_r \otimes \rho$, where ρ is restriction to H , it is also surjective. Hence, $\nu_{r+1} \circ \mu$ is surjective. This implies that $H^0(\mathbb{P}^n, \mathcal{F}(r)) = \ker \nu_{r+1} + \text{im} \mu$. Now, since the bottom row is exact, we see that

$$H^0(\mathbb{P}^n, \mathcal{F}(r)) = \text{im} \alpha + \text{im} \mu.$$

But the image of α is contained inside the image of μ , since α is obtained from μ by considering only those elements of $H^0(\mathbb{P}^n, \mathcal{O}(1))$ that are contained in the ideal of H . Hence, μ is surjective which proves (a).

(c) now follows from the fact that for some r' sufficiently large $\mathcal{F}(r')$ is globally generated and the map from $H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(r'-r)) \rightarrow H^0(\mathbb{P}^n, \mathcal{F}(r'))$ is surjective by (a) (if there was some point x

such that every global section of $\mathcal{F}(r)$ was contained in $\mathfrak{p}_x \mathcal{F}(r)$, then the same would hold for $\mathcal{F}(r')$ by surjectivity of the above map). \square

Remark. There is another useful result that can be seen from the diagram in the Lemma. Suppose ν_r as in the diagram is surjective and also suppose \mathcal{F}_H is r -regular. Then, by (a) in the Lemma, the map τ is surjective and hence ν_{r+1} is also surjective.

From the above Lemma, we see that Castelnuovo-Mumford regularity has some very useful consequences. It remains to see when this property actually holds. The following is a theorem of Mumford.

Theorem 6.4. For any non-negative integers p and n , there exists a polynomial $F_{p,n}$ in $\mathbb{Z}[x_0, \dots, x_n]$ with the following property:

Let k be any field and let \mathcal{F} be a coherent sheaf on \mathbb{P}_k^n isomorphic to a subsheaf of $\oplus^p \mathcal{O}_{\mathbb{P}^n}$. Let the Hilbert polynomial of \mathcal{F} be written in terms of binomial coefficients as

$$\chi(\mathcal{F}(r)) = \sum_{i=0}^n a_i \binom{r}{i}$$

with $a_i \in \mathbb{Z}$. Then, \mathcal{F} is m -regular, where $m = F_{p,n}(a_0, \dots, a_n)$.

Proof. This proof is pretty fun. Assume k is infinite, as before and induct on n . Also, assume \mathcal{F} to be nonzero. For $n = 0$, we can take $F_{p,n}$ to be any polynomial. So, assume $n \geq 1$. Let H be a hyperplane not containing any associated points of \mathcal{F} . By construction the sheaf

$$\underline{\text{Tor}}_{\mathcal{O}_{\mathbb{P}^n}}^1(\mathcal{O}_H, \oplus^p \mathcal{O}_{\mathbb{P}^n} / \mathcal{F}) = 0.$$

Thus, restriction to H preserves injectivity of the map $\mathcal{F} \rightarrow \oplus^p \mathcal{O}_{\mathbb{P}^n}$ and hence \mathcal{F}_H is a subsheaf of $\oplus^p \mathcal{O}_H$. Thus, we can do some induction.

Now, look at the short exact sequence

$$0 \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_H \rightarrow 0.$$

Since Euler characteristic is additive on short exact sequences, we see that

$$\chi(\mathcal{F}_H(r)) = \chi(\mathcal{F}(r)) - \chi(\mathcal{F}(r-1))$$

and hence the coefficients b_0, \dots, b_n of the Hilbert polynomial of \mathcal{F}_H satisfy

$$b_i = g_i(a_0, \dots, a_n)$$

where g_i is some polynomial in $\mathbb{Z}[x_0, \dots, x_n]$ independent of k or \mathcal{F} .

By the inductive hypothesis, there exists some $F_{p,n-1}$ in $\mathbb{Z}[x_0, \dots, x_{n-1}]$ such that \mathcal{F}_H is m_0 -regular where $m_0 = F_{p,n-1}(b_0, \dots, b_n)$. Substituting $g_j(a_0, \dots, a_n)$ we see that $m_0 = G(a_0, \dots, a_n)$ for some $G \in \mathbb{Z}[x_0, \dots, x_n]$ independent of k or \mathcal{F} . For $m \geq m_0 - 1$, the short exact sequence

$$0 \rightarrow \mathcal{F}(m-1) \rightarrow \mathcal{F}(m) \rightarrow \mathcal{F}_H \rightarrow 0$$

gives an exact sequence

$$0 \rightarrow H^0(\mathcal{F}(m-1)) \rightarrow H^0(\mathcal{F}(m)) \rightarrow H^0(\mathcal{F}_H(m)) \rightarrow H^1(\mathcal{F}(m-1)) \rightarrow 0$$

and isomorphisms $H^i(\mathcal{F}(m-1)) \rightarrow H^i(\mathcal{F}(m))$ for all $i \geq 2$. Since for some large enough m' , $H^i(\mathcal{F}(m')) = 0$, we have

$$H^i(\mathcal{F}(m)) = 0$$

for all $i \geq 2, m \geq m_0 - 2$. Additionally, the surjection $H^1(\mathcal{F}(m-1)) \rightarrow H^1(\mathcal{F}(m))$ shows that from $m_0 - 2$ onwards, $h^1(\mathcal{F}(m))$ is monotonically decreasing in m . We can actually show that it is strictly

decreasing till it hits 0. Note that equality $h^1(\mathcal{F}(m)) = h^1(\mathcal{F}(m-1))$ can hold if and only if the restriction map $H^0(\mathcal{F}(m)) \rightarrow H^0(\mathcal{F}_H(m))$ is surjective. But by the remark following Lemma 6.3, this implies that for all greater values of m , the restriction map is surjective and hence for all $m' \geq m$, $h^1(\mathcal{F}(m')) = h^1(\mathcal{F}(m))$. Thus, if equality ever holds, then at that value of m , $h^1(\mathcal{F}(m)) = 0$. Hence, we see that

$$H^1(\mathcal{F}(m)) = 0 \text{ for } m \geq m_0 + h^1(\mathcal{F}(m_0)).$$

We now derive a polynomial upper bound on $h^1(\mathcal{F}(m_0))$. Since $\mathcal{F} \subseteq \oplus^p \mathcal{O}_{\mathbb{P}^n}$, we must have

$$h^0(\mathcal{F}(m)) \leq p \binom{n+m_0}{n}.$$

Hence,

$$\begin{aligned} h^1(\mathcal{F}(m)) &= h^0(\mathcal{F}(m)) - \chi(\mathcal{F}(m)) \\ &\leq p \binom{n+m_0}{n} - \sum_{i=0}^n a_i \binom{m_0}{i} \\ &= P(a_0, \dots, a_n) \end{aligned}$$

where P is the polynomial obtained by substituting $m_0 = G(a_0, \dots, a_n)$ and is hence an element of $\mathbb{Z}[x_0, \dots, x_n]$ that does not depend on \mathcal{F} or k . Hence, we see that

$$H^1(\mathcal{F}(m)) = 0 \text{ for all } m \geq G(a_0, \dots, a_n) + P(a_0, \dots, a_n).$$

Taking $F_{p,n} = G + P + n$, and using the fact that $F_{p,n}(a_0, \dots, a_n) > m_0 + n$, we see that \mathcal{F} is $F_{p,n}(a_0, \dots, a_n)$ -regular, as desired (since we only care about cohomology upto dimension n , adding n gives us the buffer we need for regularity.) \square

Remark. Note that the above proposition also holds with subsheafs of $\oplus^p \mathcal{O}_{\mathbb{P}^n}$ replaced with quotients because if the left two terms in a short exact sequence of sheaves is m -regular, then so is the rightmost term.

We end this talk by using the above two results to prove the Proposition 5.5 that was left unproved during the construction of the Quot scheme. For convenience, we recall the proposition.

Proposition 6.5. There exists some integer m that depends only on the rank of V , the rank of W and Φ such that for all $r \geq m$, for all schemes T/S and for all T -flat coherent quotients \mathcal{F} of \mathcal{E}_{X_T} with kernel \mathcal{G} , the following facts hold:

1. $\pi_{T*}(\mathcal{F}(r)), \pi_{T*}(\mathcal{G}(r)), \pi_{T*}(\mathcal{E}_{X_T}(r))$ are locally free sheaves of ranks determined by the rank of V , rank of W , r and Φ , with $\pi_{T*}(\mathcal{F}(r))$ of rank $\Phi(r)$ in particular. In addition, all higher direct image sheaves vanish.
2. In the following commutative diagram of locally free sheaves on X_T ,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_T^* \pi_{T*}(\mathcal{G}(r)) & \longrightarrow & \pi_T^* \pi_{T*}(\mathcal{E}_{X_T}(r)) & \longrightarrow & \pi_T^* \pi_{T*}(\mathcal{F}(r)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}(r) & \longrightarrow & \mathcal{E}_{X_T}(r) & \longrightarrow & \mathcal{F}(r) & \longrightarrow & 0 \end{array}$$

the rows are exact and the vertical maps are surjective.

Proof. For any field k and any k -valued point $s \in S$, we have an isomorphism $X_s = \mathbb{P}_k^n$ with n is the rank of $V - 1$. Additionally, the restricted sheaf $\mathcal{E}|_{X_s}$ is isomorphic to $\oplus^p \mathcal{O}_{\mathbb{P}_k^n}$ where p is the rank of W . It follows from Theorem 6.4 that there exists an integer which depends only on the rank of V , the rank of W and Φ such that for any field k and any k -valued point $s \in S$, $\mathcal{E}|_{X_s}$ is m -regular and for any quotient sheaf \mathcal{F} of $\mathcal{E}|_{X_s}$, both \mathcal{F} and the kernel \mathcal{G} are m -regular. We can now apply Lemma 6.3 to see that for all $r \geq m$ and $i \geq 1$,

- (a) $H^i(X_s, \mathcal{F}(r)), H^i(X_s, \mathcal{E}_{X_s}(r)), H^i(X_s, \mathcal{G}(r)) = 0$.
- (b) $\mathcal{F}(r), \mathcal{E}_{X_s}(r), \mathcal{G}(r)$ are globally generated.

Now, suppose T is a (connected) S -scheme and \mathcal{F} is a T -flat, coherent quotient of \mathcal{E}_T with Hilbert polynomial Φ and kernel \mathcal{G} . Then, looking at the long exact sequence for Tor, we see that \mathcal{G} is also T -flat. Hence, we can apply the following results from flat base change:

Lemma 6.6. Let $\pi : X \rightarrow S$ be a proper morphism of Noetherian schemes and let \mathcal{F} be a coherent \mathcal{O}_X -module which is flat over \mathcal{O}_S . Then the following holds:

If for some integer i , there exists $d \geq 0$ such that for all $s \in S$, we have

$$\dim_{k(s)} H^i(X_s, \mathcal{F}_s) = d$$

then $R^i \pi_* \mathcal{F}$ is locally free of rank d .

Now, part 1 of the proposition holds because for $i \geq 1$ we can take $d = 0$ and for $i = 1$, we have locally freeness because the pushforwards are flat and coherent over S . The ranks are determined by the rank of V, W, r and Φ . In particular, since the Hilbert polynomial of \mathcal{F} is Φ and all higher cohomology vanishes, we have the rank of $\mathcal{F} = \Phi(r)$.

For the second part of the proposition, we need to prove that the vertical maps are surjective and the rows are exact. The bottom row is obviously exact. Surjectivity of the maps follows from global generation of the bottom row. Hence, we just need the top row to be exact. But this follows from the fact that all higher direct images of the sheaves vanish and the inverse image preserves exactness of locally free sheaves. Hence, the proposition is proved. □

REFERENCES

[Nit] Nitin Nitsure. Construction of Hilbert and Quot schemes. *arXiv:math/0504590*.