AFFINE HECKE ALGEBRAS OF TYPE A AND THEIR CYCLOTOMIC QUOTIENTS

SIDDHARTH VENKATESH

Abstract. These are notes for a talk in the MIT-NEU Graduate seminar on Hecke Algebras and Affine Hecke Algebras (AHAHA) held in Fall 2014. This talk is divided into three parts. In the first, we introduce the affine Hecke algebras and describe a useful basis for the algebra over the ground ring. We then give a complete description of the center of the affine Hecke algebra and prove Kato's Theorem regarding unique irreducibility of the Kato module in its central character block. The main reference for this part of the talk is [Kle05, 3.1-3.4, 4.1-4.3].

The second part of the talk is related to cyclotomic Hecke algebras, also called Ariki-Koike algebras in the references provided. We prove a basis theorem for these algebras and use the basis theorem to show that the cyclotomic Hecke algebras are symmetric algebras. The main references for this part of the talk is [GJ11, 5.1-5.2]. Auxilliary useful references include [AK94, BM97, MM98] and [Mac95, 1, Appendix B].

In the final part of the talk, we construct an equivalence of categories between affine Hecke algebra modules in the Kato block and modules over the center of particular character. The main references for this part of the talk is [CR04, 3.1-3.2].

Contents

Part A - Affine Hecke Algebra of Type A

1. Introduction and Notation

The Affine Hecke algebra (of Type A_n) is defined as a q-deformation of the group algebra of the affine Weyl group (of Type A_n). In this section of the talk, our goal is to construct a basis for the affine hecke

Algebra, describe the center of the algebra and then give the construction and proof of irreducibility of Kato modules, which are modules induced from the Laurent polynomial subalgebra of the affine Hecke algebra. At the end of the section, we prove the equivalence of categories given in [CR04]. We will only discuss the nondegenerate affine Hecke algebra. There are analogous results with similar proofs in the degenerate case which we will leave as an exercise (and which can be looked up in [Kle05, 3, 4].)

We fix some notation.

Notations and Conventions

(1) F denotes a commutative domain (most often a field), not necessarily of characteristic 0.
(2) For q a generic variable, A is the commutative ring $F[q^{\pm}]$ and K is its field of fractions.
(3) $\mathcal{H}_{F,n}$ (and later $\mathcal{H}_n$) denotes the affine Hecke algebra over F. $\mathcal{H}_{A,n}$ denotes the affine Hecke algebra over A with parameter equal to the polynomial variable q.
(4) We let $\mathcal{P}_{F,n}, \mathcal{P}_{A,n}, \mathcal{P}_n$ be the respective Laurent polynomial subalgebras and let $Z_{F,n}, Z_{A,n}, Z_n$ be the symmetric Laurent polynomials, which will turn out to be the centers of the affine Hecke algebras.
(5) We define $\mathcal{H}^T_{F,n}, \mathcal{H}^T_{A,n}, \mathcal{H}^T_n$ to be the (regular) Hecke subalgebra of the affine Hecke algebra.
(6) For $s \in S_n$, and $f \in \mathcal{P}_{F,n}$ we define $s \cdot f$ to be f with the variables permuted via s.

2. **Definition and Elementary Computations**

Let F be a commutative domain. Let $q \in F^{\times}$. At some point, we will assume q is not 1 (Kato Module Section) but as of now it is unimportant.

Definition 2.1. The affine Hecke algebra $\mathcal{H}_{F,n}$ (over F) is the unital associative F-algebra generated by the elements $T_1, \ldots, T_{n-1}, X_1^{\pm}, \ldots, X_n^{\pm}$ subject to

- The Eigenvalue Relations:

$$(T_i - q)(T_i + 1) = 0$$

- The Laurent Relations:

$$X_i X_j = X_j X_i$$

$$X_i X_i^{-1} = X_i^{-1} X_i = 1$$

- The Braid Relations:

$$T_i T_j = T_j T_i \text{ if } |i - j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

- The Action Relations:

$$T_i X_j = X_j T_i \text{ if } i \neq j, j - 1$$

$$T_i X_i T_i = q X_{i+1}$$

Note that the last relation also implies that $T_i X_{i+1}^{-1} T_i = qX_i^{-1}$.

We also define some useful subalgebras of $\mathcal{H}_{F,n}$.

Definition 2.2. The subalgebra generated by the X_i is denoted as $\mathcal{P}_{F,n}$.
The subalgebra of symmetric Laurent polynomials in the X_i is denoted as $Z_{F,n}$.
The subalgebra generated by the T_i is denoted as $\mathcal{H}_{F,n}^T$. In particular, if w is an element in S_n, then since the braid relations hold in $\mathcal{H}_{F,n}$, we can unambiguously define the element $T_w \in \mathcal{H}_{F,n}^T$ by taking any reduced word for w.

We end this section with the following useful computational tool. The proof is left as an exercise.

Lemma 2.3. Let $f \in \mathcal{P}_{F,n}$. We have a standard action of S_n on $\mathcal{P}_{F,n}$ by permutation of variables. If s_i denotes the transposition $(i, i+1)$, then we have

$$T_i f = (s_i \cdot f)T_i + (q-1)\frac{f - (s_i \cdot f)}{1 - X_i X_{i+1}^{-1}}$$

Now, if w is a reduced expression of an element of S_n, then, using the above relation and induction, we have

$$fT_i = T_i(s_i \cdot f) + (q-1)\frac{f - (s_1 \cdot f)}{1 - X_i X_{i+1}^{-1}}$$

and

$$T_w f = (w \cdot f)T_w + (q-1)\sum_{w'} f_{w'} T_{w'}$$

and

$$fT_w = T_w(w^{-1} \cdot f) + (q-1)\sum_{w'} T_{w'} g_{w'}$$

where the summation is over w' is a reduced expression that is contained in w i.e. over w' that are smaller than w in the Bruhat order on reduced words in S_n.

3. **Basis Theorem - Bernstein Presentation**

For $\alpha \in \mathbb{Z}^n$, define $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Our goal for this section is to prove the following theorem.

Theorem 3.1. The set

$$\mathcal{B} := \{X^\alpha T_w : \alpha \in \mathbb{Z}^n, w \in S_n\}$$

is an F-basis for $\mathcal{H}_{F,n}$.

Proof. The above lemma shows us that $F\mathcal{B}$ is invariant under left multiplication by the generators T_i, X_i and is hence all of $\mathcal{H}_{F,n}$. Thus, we need to show that $\mathcal{B}$ is linearly independent over F. It suffices to show $\mathcal{B}$ is linearly independent in $\mathcal{H}_n$, the affine Hecke algebra with generic q defined over $A = F[q, q^{-1}]$, because non-trivial relations after specializations of q can be lifted to non-trivial relations before specialization. In this case, we show that $\mathcal{B}$ is linearly independent by constructing an A-representation of $\mathcal{H}_n$ in which $\mathcal{B}$ maps to a linearly independent set of operators. Secretly, this representation is induced from the trivial representation of $\mathcal{H}_n^T$ but we do not know that until after the basis theorem is proved.

Let $\mathcal{H}_n$ act on $A[Y_1^{\pm}, \ldots, Y_n]$ via

(a)
$$X_i^{\pm} \cdot f = Y_i^{\pm} f$$

(b)
$$T_i \cdot f = s_i \cdot f + (q-1)\frac{f - s_i \cdot f}{1 - Y_i Y_{i+1}^{-1}}$$

We first show that actually defines a representation by checking that the defining relations are satisfied. It's immediate that the Laurent relations, the first action relation and the first braid relations hold. Checking that the eigenvalue relations hold is an easy computation that we leave as an exercise. So, we only need to show now that the last action relation holds and that the second braid relations hold. We compute the action relations and leave the braid relations as a similar exercise.

$$\begin{aligned}
T_i X_i T_i(f) &= T_i\left(Y_i(s_i \cdot f) + (q-1)Y_i Y_{i+1}\frac{f - (s_i \cdot f)}{Y_{i+1} - Y_i}\right) \\
&= Y_{i+1} f + (q-1)Y_{i+1}\left(\frac{Y_i(s_i \cdot f) - Y_{i+1} f}{Y_{i+1} - Y_i} + Y_i \frac{(s_i \cdot f) - f}{Y_i - Y_{i+1}}\right) \\
&\quad + (q-1)^2 \frac{Y_i Y_{i+1}}{Y_{i+1} - Y_i}\left(f - (s_i \cdot f) + (s_i \cdot f) - (s_i^2 \cdot f)\right) \\
&= Y_{i+1} f + (q-1)Y_{i+1} f = qY_{i+1} f = qX_{i+1}(f)
\end{aligned}$$

as desired.

We now check that the set $\mathcal{B}$ maps to A-linearly independent operators under this representation. Suppose we have

$$M = \sum_i c_i B_i = 0$$

with $c_i \in A$, $B_i = x^{\alpha_i} T_{w_i} \in \mathcal{B}$. We claim inductively that $c_i \in (q-1)^j \subseteq A$. The base case of $j = 0$ is obvious. Assume the statement holds for $j - 1$. Then, for any $f \in A[Y_1^{\pm}, \ldots, Y_n^{\pm}]$, the action of M (modulo $(q-1)^j$) is given by

$$0 = \sum_i c_i x^{\alpha_i}(w_i \cdot f).$$

Now, if we let N be bigger than all the $|\alpha_i|$ (which is the sum of the absolute value of its components), then choosing $f = X_1^N \cdots X_n^{nN}$, we see that the monomials

$$x^{\alpha_i}(w_i \cdot f) = x^{\alpha_j}(w_j \cdot f) \Leftrightarrow i = j.$$

Hence, using both these relations, we see that $c_i \equiv 0 \mod (q-1)^j$, which completes the induction step.
Hence, we have

$$c_i \in \cap_{j=0}^{\infty}(q-1)^j = 0.$$

Thus, the set $\mathcal{B}$ is linearly independent over A, and hence forms a basis, as desired.

□

We have an obvious corollary of the above theorem.

Corollary 3.2. $\mathcal{P}^{F,n}$ is isomorphic to the algebra of Laurent polynomials in n variables over F.
$\mathcal{H}_{F,n}^{T}$ is isomorphic to the Hecke algebra of type A_{n-1} over F.

This completes this section. We now move on to describing the center of the affine Hecke algebra.

4. **The Center of the Affine Hecke Algebra**

Let $Z_{F,n}$ be the sualgebra of symmetric Laurent polynomials in the X_i. Then, we have the following result.

Theorem 4.1. $Z(\mathcal{H}_{F,n}) = Z_{F,n}$.

Proof. Note that $Z_{F,n}$ clearly commutes with each X_i and it also commutes with each T_i by Lemma 2.3. Hence, $Z_{F,n} \subseteq Z(\mathcal{H}_{F,n})$.
To get the reverse inclusion, we first show that $Z(\mathcal{H}_{F,n}) \subseteq F[X_1^{\pm}, \ldots, X_n^{\pm}]$. We use the basis theorem proved before. Suppose

$$f = \sum_i c_i X^{\alpha_i} T_{w_i} \in Z(\mathcal{H}_{F,n}).$$

Let w_0 in the decomposition of f be an element of S_n that is maximal in the Bruhat order. Suppose for contradiction that $f \notin \mathcal{P}_{F,n}$. Then, $w_0 \neq e$. Let $j \in \{1, \ldots, n\}$ be such that $w_0(j) \neq j$. Then,

$$X_j f = \sum_i c_i X_j X^{\alpha_i} w_i,$$

which has w_0 coefficient $c_0 X_0 X^{\alpha_0}$, but by Lemma 2.3, the w_0 coefficient of

$$f X_j = \sum_i c_i X^{\alpha_i} w_i X_j$$

is $c_0 X_{w_0(j)} X^{\alpha_0}$ which is not the same. Hence, by the basis theorem, $X_j f \neq f X_j$, a contradiction. Hence, $Z(\mathcal{H}_{F,n}) \subseteq \mathcal{P}_{F,n}$.

We now show that $Z(\mathcal{H}_{F,n}) \subseteq Z_{F,n}$. By Lemma 2.3, for a Laurent polynomial f,

$$T_i f = (s_i \cdot f) T_i + (q-1) \frac{f - (s_i \cdot f)}{1 - X_i X_{i+1}^{-1}}.$$

So, if $f \in Z(\mathcal{H}_{F,n})$, then, by the basis theorem,

$$f T_i = (s_i \cdot f) T_i$$

and hence f is symmetric (as i is arbitrary).

□

5. **Kato Modules and Kato's Theorem**

From now on, let F be a field and let $q \in F^\times$ not be 1. Let $\mathcal{H}_{F,n}$ be denoted by $\mathcal{H}_n$ (analogous convention for $\mathcal{P}_n, \mathcal{H}_n^T, Z_n$). Let $\mathcal{P}_n$-mod be the category of finite-dimensional (over F) $\mathcal{P}_n$ modules. We want to understand the $\mathcal{H}_n$-modules of particular central character i.e. modules in which $\mathbb{Z}_n$ acts by each X_i acting as the same scalar. The last section of this talk will give a complete description of such modules. In this section, we merely study the simple objects in this subcategory of $\mathcal{H}_n$-mod. Our goal of this section is thus to prove the following theorem.

Theorem 5.1. Let $a \in \mathbb{F}^\times$. Take the one-dimension $\mathcal{P}_n$-module $L(a, \dots, a)$ in which X_i acts as a and define $L(a^n)$ to be the induced $\mathcal{H}_n$-module

$$\operatorname{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(L(a, \dots, a)).$$

Then, $L(a^n)$ is the unique simple module in its central character block.

We first introduce the notion of formal characters on $\mathcal{P}_n$-modules.

Definition 5.2. For $\underline{a} = (a_1, \dots, a_n) \in (F^\times)^n$, we have a one-dimensional representation of $\mathcal{P}_n$ in which X_i acts as a_i. These form a complete list of irreducible $\mathcal{P}_n$-modules.

For an arbitrary finite-dimensional $\mathcal{P}_n$-module M, let $M_{\underline{a}}$ be the largest submodule whose composition factors are $L(\underline{a})$ or equivalently, the generalized eigenspace in M in which X_i has eigenvalue a_i.

We then have the obvious result.

Lemma 5.3. For any $M \in \mathcal{P}_n$-mod, we have

$$M = \oplus_{\underline{a} \in (F^\times)^n} M_{\underline{a}}.$$

We now define the formal character for a representation $M \in \mathcal{H}_n$-mod.

Definition 5.4. For $M \in \mathcal{P}_n$-mod, let $[M]$ be its image in the Grothendieck group. Then, for $M \in \mathcal{H}_n$-mod, we define the formal character of M, $\operatorname{ch} M$, to be $[\operatorname{Res}_{\mathcal{P}_n} M]$.

Since $\operatorname{Res}_{\mathcal{P}_n}$ is an exact functor, ch is a homomorphism from the Grothendeick group of $\mathcal{H}_n$-mod to the Grothendeick group of $\mathcal{P}_n$-mod. The following Lemma will be useful later.

Lemma 5.5. Let $\underline{a} = (a_1, \dots, a_n) \in (F^\times)^n$. Then,

$$\operatorname{ch} \operatorname{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n} L(\underline{a}) = \sum_{w \in S_n} [L(w(\underline{a}))]$$

where $w(\underline{a})_i = a_{w^{-1}(i)}$.
In particular, note that

$$\operatorname{ch} \operatorname{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(L(a, \dots, a)) = n! L(a, \dots, a).$$

Proof. Fix $\underline{a} = (a_1, \dots, a_n)$. By the basis theorem, an F-basis for $L(\underline{a})$ is given by $\{T_q \otimes 1 : w \in S_n\}$. Put a total order on this basis by refining the Bruhat order on S_n. Then, for each i from 1 to n, we have by Lemma 2.3

$$X_i(T_w \otimes 1) = (T_w \otimes X_{w^{-1}(i)}(1)) + \sum_{w' < w} T_{w'} \otimes g_{w'}(1) = a_{w^{-1}(i)}(T_w \otimes 1) + \sum_{w' < w} T_{w'} \otimes g_{w'}(1).$$

Hence, every X_i acts as an upper triangular matrix in the given basis for $L(a^n)$ with a's on the diagonals. Now, we have an ascending filtration $\{\mathcal{M}_l\}$ (as a $\mathcal{P}_n$-mod) of $\mathrm{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n} L(\underline{a})$ where

$$\mathcal{M}_l = \bigoplus_{i=1}^{l} F(T_{w_i} \otimes 1).$$

Thus, since the filtration and its associated graded module give the same element in the Grothendieck group, we have by the above computation,

$$\mathrm{ch}\,\mathrm{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(L(\underline{a})) = \bigoplus_{i=1}^{n!} \mathrm{ch}(\mathcal{M}_i/\mathcal{M}_{i-1}) = \bigoplus_{i=1}^{n!} L(w_i(\underline{a}))$$

as desired.

□

We now define central characters of representations in $\mathcal{H}_n$-mod.

Definition 5.6. Since $Z(\mathcal{H}_n) = Z_n$, for each $\underline{a} = (a_1, \ldots, a_n) \in (\mathbb{F}^\times)^n$ we can define a homomorphism

$$\chi_{\underline{a}} Z_n \to F$$

by sending the Laurent polynomial $f(x_1, \ldots, x_n) \mapsto f(a_1, \ldots, a_n)$.
By elementary theory of symmetric functions, we have $\chi_{\underline{a}} = \chi_{\underline{b}}$ if and only if $\underline{b}$ is in the orbit of $\underline{a}$ in the action of S_n. We say that $\chi_{\underline{a}}$ is a central character of $\mathcal{H}_n$. If γ is the orbit in F^n of α under the action of S_n, we also say that γ is a central character of $\mathcal{H}_n$.

Now, in any irreducible in $\mathcal{H}_n$-mod, Z_n acts via a particular central character. Thus, $\mathcal{H}_n$-mod splits up as the direct sum of abelian subcategories corresponding to a particular central character. This gives us the following definition.

Definition 5.7. The subcategory of $\mathcal{H}_n$-mod in which every object has, as composition factors, irreducibles in which the center Z_n acts via the character γ is called the block in $\mathcal{H}_n$-mod corresponding to γ. We denote the block corresponding to γ by $\mathcal{H}_n$-mod$[\gamma]$.

We are now ready to define Kato modules and prove Kato's theorem regarding these modules. Kato modules are objects that are analogous to Verma modules in Lie theory.

Definition 5.8. For $a \in F^\times$, we define the Kato module (with central character $\gamma_a = (a, \ldots, a)$) to be

$$L(a^n) := \mathrm{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(L(a, \cdots a)).$$

By Lemma 5.5, we know that $L(a^n)$ has central character γ_a and hence belongs to the block corresponding to this character. Kato's Theorem then states that $L(a^n)$ is the unique irreducible in its block.

We build up to Kato's Theorem by first proving the following lemma.

Lemma 5.9. Let $a \in F^\times$. Let $L = L(a, \cdots, a)$ (and hence $L(a^n) = \mathcal{H}_n \otimes_{\mathcal{P}_n} L$). The common a-eigenspace of the operators $X_1, \ldots, X_{n-1}$ on $L(a^n)$ is precisely $1 \otimes L$.

Proof. By the basis theorem, $L(a^n) = \oplus_{w\in S_n} T_w \otimes L$. We prove by induction that the common a-eigenspace for $X_1, \ldots, X_i$ is

$$\bigoplus_{y\in\langle s_{i+1},\ldots s_{n-1}\rangle} T_y \otimes L.$$

We denote $\langle s_{i+1}, \ldots, s_{n-1}\rangle$ as Δ_i. We use the base case of $i = 0$, which is vacuously true. So, now suppose that $i \geq 1$ and suppose that the common a-eigenspace for $X_1, \ldots, X_{i-1}$ is

$$\bigoplus_{y\in\Delta_{i-1}} T_y \otimes L.$$

Now, any T_w for $w \in \Delta_{i-1}$ can be written as $T_{w'}T_iT_{i+1}\cdots T_j$ with $i-1 \leq j \leq n-1$ and $w' \in \Delta_i$. Then, by Lemma 2.3 and some computation, we have for any $v \in L$,

$$(X_i - a)(T_w \otimes v) = -(q-1)aT_{w'}T_i \cdots T_{j-1} \otimes v + (*)$$

where $(*)$ stands for terms that belong to

$$\bigoplus_{y'\in\Delta_i,k<j-1} T_{y'}T_i \cdots T_k \otimes L.$$

Now, suppose z is in the common a-eigenspace of $X_1, \ldots, X_i$. By the induction hypothesis, we can write, for some fixed nonzero $v \in L$,

$$z = \sum_{y\in\Delta_{i-1}} c_yT_y \otimes v = \sum_{w'\in\Delta_i,i-1\leq j\leq n-1} c_{w',j}T_{w'}T_i \cdots T_j \otimes v.$$

Choose maximal j for which $c_{w',j}$ is nonzero (for some w'). But then, the above calculation shows that, since $q \neq 1$ unless $j = i-1$ (i.e. there are only the w' terms), $(X_i - a)z$ has a nonzero $T_{w'} \cdots T_{j-1}$ coefficient. Thus, by the basis theorem, we have $z \in \bigoplus_{y\in\Delta_i} T_y \otimes L$. This completes the induction step. Since $\Delta_{n-1} = \{1\}$, we have the desired result.

□

We finish the section by proving Kato's Theorem.

Theorem 5.10. Let $a \in F^\times$ and let $\mu = (\mu_1, \ldots, \mu_r)$ be a composition of n. We define $\mathcal{H}_\mu = H_{\mu_1} \otimes \cdots \otimes H_{\mu_r}$ and define $\mathcal{H}_{n-1}$ to be the affine Hecke subalgebra generated by $X_1^\pm, \ldots, X_{n-1}^\pm$ and $T_1, \ldots, T_{n-2}$. Then:

(1) $L(a^n)$ is irreducible and it is the only irreducible in its block.

(2) All composition factors of $\mathrm{Res}_{\mathcal{H}_\mu}(L(a^n))$ are isomorphic to

$$L(a^{\mu_1}) \boxtimes \cdots \boxtimes L(a^{\mu_r})$$

and the socle of $\mathrm{Res}_{\mathcal{H}_\mu}(L(a^n))$ is irreducible.

(3) The socle (the sum of all simple submodules) of $\mathrm{Res}_{\mathcal{H}_{n-1}}(L(a^n)) \cong L(a^{n-1})$.

Proof. As before, let L denote $L(a, \cdots a) \in \mathcal{P}_n$-mod.

(1) Let M be a nonzero $\mathcal{H}_n$-submodule of $L(a^n)$. Since $L(a^n)$ restricted to $\mathcal{P}_n$ has composition factors all isomorphic to L, so does M by Lemma 5.5. Hence, $\mathrm{Res}_{\mathcal{P}_n}(M)$ contains a $\mathcal{P}_n$-submodule N isomorphic to L. Now, $\mathcal{P}_n$ acts on L via scalars in which each X_i acts as a. Thus, N is contained in $1 \otimes L$, the common a-eigenspace of $X_1, \ldots, X_n$. But, we know that $1 \otimes L$ is irreducible as a $\mathcal{P}_n$-module. Hence, $N = 1 \otimes L$ and hence

$$M \supseteq \mathcal{H}_n(1 \otimes L) = L(a^n).$$

Thus, $L(a^n)$ is irreducible. Now, if M' is any other representation in the same block, by central character considerations, $\mathrm{Res}_{\mathcal{P}_n}(M')$ must contain a $\mathcal{P}_n$-submodule isomorphic to L and hence by Frobenius Reciprocity, M' contains an $\mathcal{H}_n$-submodule isomorphic to $L(a^n)$.

(2) The fact that all composition factors of $\mathrm{Res}_{\mathcal{H}_\mu}(L(a^n))$ are isomorphic to $L(a^{\mu_1}) \boxtimes \cdots \boxtimes L(a^{\mu_r})$ is immediate from unicity of irreducibility of $L(a^{\mu_i})$ and central character considerations via Lemma 5.5. To see that the socle of $\mathrm{Res}_{\mathcal{H}_\mu}(L(a^n))$ is irreducible first note that the $\mathcal{H}_\mu$-submodule $\mathcal{H}_\mu L \cong H_\mu \otimes L$ of $L(a^n)$ is isomorphic to the irreducible

$$L(a^{\mu_1}) \boxtimes L(a^{\mu_r}).$$

Conversely if M is any irreducible $\mathcal{H}_\mu$-submodule of the restriction, then using the same argument as in the proof of (1), we see that M contains L and hence must be $H_\mu L$. Thus, $H_\mu \otimes L$ is the socle, which is hence irreducible.

(3) First note that by part 2, $L(a^n)$ has a unique $\mathcal{H}_{n-1,1}$ submodule $\mathcal{H}_{n-1,1} \otimes L \cong L(a^{n-1}) \boxtimes L(a)$, which is the socle of the restriction of $L(a^n)$. This gives us a one-dimensional contribution of $L(a^{n-1})$ to the socle. However, if M is any other irreducible $\mathcal{H}_{n-1}$-submodule of $L(a^n)$, then M must contain $\mathcal{H}_{n-1} \otimes L$ by the same argument as in (1). Hence, the socle of $\mathrm{Res}_{\mathcal{H}_{n-1}}(L(a^n))$ is isomorphic to $L(a^{n-1})$.

□

Part B - Cyclotomic Hecke Algebras of Type A

6. Introduction and Notation

Cyclotomic Hecke Algebras, called Ariki-Koike algebras in the main reference [GJ11], are defined as q-deformations of the group algebras of the complex reflection groups of type $G(m, 1, n)$. They can also be viewed as cyclotomic quotients of the affine Hecke algebra. We begin Part B by briefly describing the construction and representation theory of these groups.

Additionally, we define some notation here that will be fixed for the section of the talk on AK-Algebras. This list of notation is for the convenience of the reader. Terms used in the notation section will be defined when introduced later.

Notations and Conventions

(1) R denotes an arbitrary commutative domain with unity of characteristic 0, unless specified otherwise. k is the field of fractions of R.
(2) For $q, q_1, \ldots, q_{n-1}$ generic variables, A is the commutative ring $R[q^{\pm}, q_1, \ldots, q_{n-1}]$ and K is its field of fractions.
(3) We fix an $m \in \mathbb{Z}_{>0}$. Then, W_n denotes the complex reflection groups of type $G(m, 1, n)$.
(4) $H_{R,n}$ denotes the Ariki-Koike Algebra associated to $G(m, 1, n)$ over the ring R.
$H_{A,n}$ denotes the Ariki-Koike algebra associated to $G(m, 1, n)$ over the ring A, with generic q, q_i.
H_n denotes $K \otimes_A H_{A,n}$.
(5) For λ a partition of n, we write $\lambda \vdash n$.
For λ an m-partition of n, we write $\lambda \vdash_m n$.

7. **Complex Reflection Groups of Type** $G(m, 1, n)$

Definition 7.1. A complex reflection in $GL(r, \mathbb{C})$ is a matrix whose 1-eigenspace has dimension $r - 1$. In other words, a complex reflection is an automorphism of a complex vector space which fixes some hyperplane pointwise. Note however, that a complex reflection does not have to have order 2.
A complex reflection group is a subgroup of $GL(r, \mathbb{C})$ for some r that is generated by complex reflections.

Finite complex reflection groups have been completely classified by Sheppard and Todd. In their classification, there are 34 exceptional groups and one infinite family of groups $G(m, p, n)$, where m, p, n are positive integers. In this talk, we will only care about the complex reflection groups of type $G(m, 1, n)$. We now give 3 realizations of this group:

1. $G(m, 1, n)$ is the wreath product $\mathbb{Z}/m\mathbb{Z} \wr S_n$, which is the group $(\mathbb{Z}/m\mathbb{Z})^n \rtimes S_n$ with the symmetric group acting via permutation of coordinates.
2. We can also define $G(m, 1, n)$ using generators and relations. There is a presentation of $G(m, 1, n)$ with generators $S = \{s_i : i = 0, \ldots, n-1\}$ and relations
   - $s_0^m = 1$
   - $s_i^2 = 1$ for $i > 0$.
   - $s_i s_j = s_j s_i$ if $|i - j| > 1$.
   - $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.
   - $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ if $i \geq 1$.

   The last 3 relations are called the braid relations in $G(m, 1, n)$.
3. Finally, we can realize $G(m, 1, n)$ inside $GL(n, \mathbb{C})$ as the subgroup of monomial matrices with entries that are mth roots of unity. Here, for some primitive mth root of unity ζ_m, we have $s_0 = \zeta_m E_{1,1} + \sum_{j \neq i} E_{jj}$ and $s_i = E_{i,i+1} + E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj}$.
   It is this last definition that makes it clear that $G(m, 1, n)$ is a finite complex reflection group.

From now on, fix a positive integer m and let W_n denote $G(m, 1, n)$. As a useful fact, we note that $|W_n| = m^n n!$.

We now define some combinatorial objects that generalize the notion of a partition and that will be useful in the representation theory of W_n and the associated AK-algebra.

Definition 7.2. We call $\lambda = (\lambda^1, \ldots, \lambda^m)$ an m-partition of n, if each λ^i is a partition of $\{\lambda_{i-1}+1, \ldots \lambda_{i-1}+\lambda_i\}$ and $\sum_{i=1}^m |\lambda^i| = n$.
Note that in m-partitions, we allow some of the λ^i to have size 0.

Let R now be a commutative domain of characteristic 0 and let k be its fraction field. Assume that R contains the mth roots of unity. Then, it turns out that k is a splitting field for W_n and that the irreducible representations of W_n over k are indexed by m-partitions of n. We give a brief description of these representations and leave the verification of the details as an exercise that can be looked up in [Mac95, 1, Appendix B] if needed.

First, we define m 1-dimensional representations of W_n via the m irreducible characters of $\mathbb{Z}/m\mathbb{Z}$. Let ζ_m be a fixed primitive mth root of unity. We then define the representation σ_k by sending

$$s_0 \mapsto \zeta_m^k \in k$$

and

$$s_i \mapsto 1, i > 0.$$

Let $\lambda = (\lambda^1, \ldots, \lambda^m)$ now be an m-partition of n and suppose $n_i = |\lambda^i|$.. For each i, we can use the natural projection $W_{n_i} \to S_{n_i}$ to extend the irreducible Specht module E^{λ_i} of S_{n_i} to an irreducible representation of W_{n_i}. Then, since we have $W_n^\lambda = W_{n_1} \times \cdots W_{n_m} \subseteq W_n$, we can define the representation

$$E^\lambda := \mathrm{Ind}_{W_n^\lambda}^{W_n}((E^{\lambda^1} \otimes \sigma_1) \boxtimes (E^{\lambda^2} \otimes \sigma_2) \boxtimes \cdots \boxtimes (E^{\lambda^m} \otimes \sigma_m)).$$

We now have the following theorem:

Theorem 7.3. The above procedure gives a complete list of non-isomorphic simple W_n representations, that is,

$$\mathrm{Irr}_k(W_n) = \{E^\lambda : \lambda \vdash_m n\}.$$

We next describe the branching rule for restriction of representations from W_n to W_{n-1}. To do so, we generalize the notion of Young Tableaux to m-partitions.

Definition 7.4. For $\lambda = (\lambda^1, \ldots, \lambda^m) \vdash_m n$, we define the Young tableau $[\lambda]$ of λ to be an m-tuple of Young tableaux $([\lambda^1], \ldots, [\lambda^m])$.
We define the set of addable (resp. removable) boxes, $\mathrm{add}(\lambda)$ (resp. $\mathrm{rem}(\lambda)$), to be the union of the set of addable (resp. removable) boxes in each component tableau.
For $x \in \mathrm{add}(\lambda)$ (resp. $\mathrm{rem}(\lambda)$), we define $[\lambda + \{x\}]$ (resp. $\lambda - \{x\}]$) as the m-partition obtained by adding (resp. removing) the box x.

Then, we have the following branching rule in W_n:

Theorem 7.5. For all $\lambda \vdash_m n$, we have

$$\mathrm{Res}_{W_{n-1}}^{W_n}(E^\lambda) = \oplus_\mu E^\mu$$

where the sum is taken over all $[\mu]$ that are obtained by removing a box from $[\lambda]$.

This finishes our discussion of the representation theory of the complex reflection groups W_n. We now move on to the associated Cyclotomic Hecke Algebra.

8. Cyclotomic Hecke Algebra: Definition and Examples

From now on, we fix the commutative domain R containing $\mathbb{C}$ and let k be its field of fractions. We define the cyclotomic Hecke algebra $H_{R,n}$ as a deformation of the group algebra $R[W_n]$. Let $q, q_1, \ldots, q_m \in R^\times$. Then:

Definition 8.1. The CH-algebra $H_{R,n} = H_{R,n}(q, q_1, \ldots, q_m)$ is defined as the unital associative R-algebra generated by the elements $T_0, \ldots, T_{n-1}$ subject to

- The Eigenvalue relations:

$$(T_0 - q_1) \cdots (T_0 - q_m) = 0$$

$$(T_i - q)(T_i + 1) = 0 \text{ for } i > 0.$$

- The Braid Relations:

$$T_i T_j = T_i T_j \text{ if } |i - j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } i > 0$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0.$$

We give a few examples of $H_{R,n}$ for particular values of q, q_i.

Example 8.2. We give 3 examples here and leave the verification of the details as an exercise.

1. Suppose R contains the primitive mth root of unity ζ_m. If $q = 1$ and $q_j = \zeta_m^j$. Then, $H_{R,n} \cong R[W_n]$. In particular, if instead of R, we use its field of fractions k, then the corresponding Ariki-Koike algebra is split semisimple.
2. Suppose $l = 1$ and suppose q has a square root in R. Then, $H_{R,n}$ is the Hecke algebra over R of type A_{n-1}.
3. Suppose $l = 2$ and suppose q, q_1, q_2 have square roots in R. Then, $H_{R,n}$ is the Hecke algebra over R of type B_n.

So we see that many interesting algebras are simply special cases of $H_{R,n}$, which gives us enough reason to want to understand its structure and representation theory.

We end the section with the following important remark:

Remark. There is a very useful realization of the cyclotomic Hecke algebra as a quotient of the affine Hecke algebra $\mathcal{H}_{R,n}$ by the two sided ideal generated by $(X_1 - q_1) \cdots (X_1 - q_m)$. From the definition of the defining relations for each algebra, it is not difficult to see that there is a surjective algebra homomorphism

$$\Phi : \mathcal{H}_{R,n} \to H_{R,n}$$

given by sending T_i to T_i for $i > 0$ and sending X_1 to T_0. This gives additional motivation to the study of cyclotomic Hecke algebras because any finite dimensional representation of the affine Hecke algebra factors through a cyclotomic quotient.

9. **Jucys-Murphy Basis of Cyclotomic Hecke Algebras**

We now define some special elements of $H_{R,n}$ that are called Jucys-Murphy elements. For $j = 1, \ldots, n$, define

$$L_j = q^{1-j} T_{j-1} \cdots T_1 T_0 T_1 \cdots T_{j-1}.$$

Remark. If we assume $m = 1$ and specialize at $q = 1$, then $H_{k,n} \cong k[S_n]$. However, under this specialization, the Jucys-Murphy elements defined here are not the same as the Jucys-Murphy elements defined classically.

Remark. The Jucys-Murphy elements defined above can also be defined as $L_j = \Phi(X_j)$, where Φ is the map defined in the terminal remark of the previous section. All of the following identities for the Jucys-Murphy elements can thus be proved by proving them at the level of the affine Hecke algebra.

We note down some useful properties of the Jucys-Murphy elements and leave the proofs as an exercise (in applying the braid relations or using the affine Hecke algebra relations).

Proposition 9.1. For L_i defined as above, we have:

1. L_i commutes with L_j.
2. T_i commutes with L_j if $j \neq i, i+1$.
3. T_i commutes with $L_i L_{i+1}$ and $L_i + L_{i+1}$.
4. For all $a \in R$ and $i \neq j$, T_i commutes with $\prod_{1 \leq l \leq j} (L_l - a)$.

Now, as the generators s_i for $i > 1$ satisfy the relations of the Hecke algebra of type A_{n-1}, we can uniquely define T_w for any $w \in S_n$. Then, as a corollary of the proposition above, we have the following Lemma.

Lemma 9.2. The following identities hold in $H_{R,n}$:

(1) For $i \geq 1$,

$$T_i L_{i+1}^k = q^k L_i^k T_i + (q-1) \sum_{j=1}^{k} q^{1-j} L_{i-1}^{j-1} L_i^{k-j+1}.$$

(2) For $i \geq 1$,

$$T_i L_i = q^{-k} L_{i+1}^k T_i + (q^{-1} - 1) \sum_{j=1}^{k} q^{1-j} L_i^{k-j} L_{i+1}^{j}.$$

We leave the proof of the Lemma as an exercise. We note that the exact formulas are not very important. The key idea is that T_i applied to either L_{i+1}^k or L_i^k interpolates between L_{i+1} and L_i while keeping the total power constant.

We now define a distinguished set $X \subseteq H_{R,n}$ of size $m^n n!$ as follows.

Definition 9.3.

$$X := \{L_1^{c_1} \cdots L_n^{c_n} T_w : w \in S_n, 0 \leq c_i \leq m-1\}.$$

Our goal in this section is to prove that X is a basis for $H_{R,n}$ over R. With Lemma 9.2 in hand, we can prove the first part of this goal as the following theorem.

Theorem 9.4. The set X spans $H_{R,n}$ over R.

Proof. Since $R\langle X\rangle$ contains the unit element, it suffices to show that this set is stable under left multiplication by $H_{R,n}$. For this, it suffices to show that $R\langle X\rangle$ is stable under left multiplication by each T_i. Fix some $0 \leq c_1, \ldots, c_n \leq m-1$ and some $w \in S_n$. Let c be the m-tuple $(c_1, \ldots, c_m)$ and let $L_{c,w}$ denote the obvious Jucys-Murphy element. Then, we have

$$T_0 L_{c,w} = L_{c+1,w}.$$

For $c < m-1$, this gives another element of X. For $c = m-1$, we can use the Eigenvalue relation to write this as a sum of $L_{a,w}$ with $0 \leq a \leq m-1$. Hence, $R\langle X\rangle$ is stable under left multiplication by T_0. Now, fix some $0 < i < n$. Then, Proposition 9.1 implies that

$$T_i L_{c,w} = L_1^{c_1} \cdots, T_i L_i^{c_i} L_{i+1}^{c_{i+1}} \cdots L_n^{c_n} T_w.$$

Hence, again using the same proposition, it suffices to show that for arbitrary $0 \leq u, v < l$, we can write $T_i L_i^u L_{i+1}^v$ as a sum of $L_i^a L_{i+1}^b T_i$ and $L_i^{a'} L_{i+1}^{b'}$ with $a, b, a', b' < l$. We prove this fact in the case with $u > v$. The proof in the other case follows very similarly. So, assume $u > v$. Then, using Proposition 9.1 and Lemma 9.2, we have for $X = (L_i L_{i+1})^v$,

$$T_i L_i^u L_{i+1}^v = T_i L_i^{u-v} X = L_{i+1}^{u-v} X T_i + \sum_{j=1}^{u-v} L_i^{u-v-j} L_{i+1}^j X$$

with the scalars suppressed in the above equation. Since L_i, L_{i+1} commute, we have the desired expression.

□

We can now state some corollaries of the above theorem.

Corollary 9.5. Over any field $\mathbb{K}$, $H_{\mathbb{K},n}$ has dimension at most $n^m n!$

Corollary 9.6. To prove that X is linearly independent over R, it suffices to show linear independence in the generic case i.e. to prove that X is linearly independent in $H_{A,n}$ where $A = R[q_1^{\pm}, \ldots, q_m^{\pm}, q^{\pm}]$ is the polynomial ring in $q_i^{\pm}, q^{\pm}$ over R and the parameters for the cyclotomic Hecke algebra are chosen to be the generic variables.

Proof. Let $q^{\pm}, q_1, \ldots, q_m$ now be indeterminate variables and let $A = R[q^{\pm}, q_1, \ldots, q_m]$. Suppose, for some $X_1, \ldots, X_l \in X$ we have

$$\sum_i c_i X_i = 0$$

with $c_i \in R$ not all 0, where we view R as an arbitrary specialization of A. Let b_i be a lift of c_i in A. Then, if q, q_j specialize respectively to ϵ, ϵ_j, then we have

$$\sum_i b_i X_i \in A(q-\epsilon, q_j - \epsilon_j)A.$$

But since the elements of X span $H_{A,n}$ over A (since nothing special about R was used in the previous theorem), we can rewrite the above relation as the equation

$$\sum_i b_i X_i = \sum_{i'} d_{i'} X_{i'}$$

which gives us a nontrivial relation over A (it's non-trivial because the $d_{i'}$ must map to 0 under specialization so they can't all be the same as the b_i.) Thus, it suffices to prove linear independence in the generic case i.e. over A. □

Let A now be defined in the above corollary and let K be its field of fractions. Proving linear independence of X over A is the same as proving linear independence of X over K. Thus, we let H_n now denote $H_{K,n}$ and we prove the statement of linear independence by explicitly constructing a large enough set of irreducible representations of H_n over K and then using a dimension counting argument.

9.1. **Irreducible Representations of H_n.** The full details of the construction of the irreducible representations of H_n is purely technical and is hence left to the appendix of the notes. Here, we merely highlight the salient details of the construction.

Definition 9.7. Let $\lambda \vdash_m n$. We define a standard Young tableau of shape λ to be an enumeration from 1 to n of the boxes of the Young diagram of λ such that each component tableaux is enumerated in a standard manner i.e increasing in rows and columns. For a fixed λ, define V_λ be the formal K-linear span of all standard Young tableaux of shape λ.

It is possible to define an action of H_n on V_λ. The full details of this action are left to the appendix. The main properties that we will use are:

(1) For a standard Young tableau P of shape λ, T_0 acts as the scalar u_i where i the index of the component of P in which 1 appears.
(2) Fix a standard Young tableau P of shape λ. For $i > 1$, if swapping i and $i+1$ in P does not give a valid standard Young tableau, then T_i acts as a nonzero scalar.

    If swapping i and $i+1$ in P gives a standard Young tableau Q, then $T_i(P)$ is a linear combination of P and Q, with nonzero Q coefficient.

We now come to the main result of this section.

Theorem 9.8. The above action of H_n on V_λ gives V_λ the structure of an absolutely irreducible representation of H_n. Additionally, if $\lambda \neq \mu$, then $V_\lambda \ncong V_\mu$ as representations of H_n.

Proof. Verifying that each V_λ is actually a representation of H_n is a tedious exercise in checking that the eigenvalue and braid relations hold. This can be looked up in [AK94, Thm 3.7]. We assume this fact here and just prove the absolute irreducibility and inequivalence of V_λ for distinct λ. For this, we induct on n.

The base case of $n = 1$ is obvious. So, we assume the statement holds for $n-1$. We first prove absolute irreducibility of V_λ i.e. that $V_\lambda \otimes \overline{K}$ is irreducible. Note that $T_0, \ldots, T_{n-2}$ generate a subalgebra G_{n-1} of H_n that is isomorphic to a quotient subalgebra of H_{n-1}. Hence, via restriction and pullback, we can view $V_\lambda \otimes \overline{k}$ as a representation of H_{n-1}.

Now, suppose there are l removable boxes in the diagram corresponding to λ and let $V_\lambda^{(i)}$ be the subspace of V_λ whose basis is given by the standard Young tableaux in which n is in removable box i. Then, as a representation of H_{n-1}, V_λ breaks up as

$$V_\lambda \otimes \overline{K} = \bigoplus_{i=1}^{l} V_\lambda^{(i)} \otimes \overline{K}$$

because $T_0, \ldots, T_{n-2}$ do not affect the position of n. But, if μ is the m-partition of $n-1$ which is obtained by removing removable box i from λ, then, since the action of T_i for $i < n-1$ does not depend on the position of n, tha as representations of H_{n-1}, we have

$$V_\lambda^{(i)} \cong V_\mu.$$

Thus, by induction, each $V_\lambda^{(i)} \otimes \overline{K}$ is irreducible and for distinct i, j, the corresponding irreducibles are nonisomorphic. So, now, let W be a nonzero H_n subrepresentation of $V_\lambda \otimes \overline{K}$. By restricting to G_{n-1}, we see that W must contain some $V_\lambda^{(i)} \otimes \overline{K}$. We need to show that it contains all $V_\lambda^{(j)} \otimes \overline{K}$, and to do so, by distinct irreducibility over G_{n-1} of the latter, we note that it suffices to show that W intersects each $V_\lambda^{(j)} \otimes \overline{K}$ nontrivially.

Pick some standard Young tableau T_ρ of shape λ such that n is in removable box i and such that $n-1$ is in removable box $j \neq i$. Let $T_{\rho'}$ be the SYT with n, $n-1$ swapped. Then,

$$T_{n-1}(T_\rho \otimes 1) = \lambda_1 T_\rho + \lambda_2 T_{\rho'}$$

with $\lambda_2 \neq 0$. This implies that $T_{\rho'} \otimes 1 \in W$. Hence, for each j, W intersects $V_\lambda^{(j)} \otimes \overline{K}$ nontrivially and hence contains it. Thus, $W = V_\lambda \otimes \overline{K}$, which is hence irreducible.

All that remains is to show that $V_\lambda \ncong V_\eta$ is $\lambda \neq \eta$. But this follows, after restricting to G_{n-1}, from the fact that the diagrams obtained by removing one box from λ is not the same set as the diagrams obtained by removing one box from η unless $\lambda = \eta$. □

This theorem now completes the basis theorem that we desired.

Theorem 9.9. The subset X of H_n defined above as

$$X = \{L_1^{c_1} \cdots L_n^{c_n} T_w : w \in S_n, 0 \leq c_i \leq m-1\}$$

is Linearly Independent over K. Additionally, H_n is semisimple.

Proof. Since we know that X spans H_n over K, for the set to be linearly independent, it suffices to show that the K-dimension of X is at least $m^n n!$. If f_λ is the dimension of the irreducible V_λ, then we know that

$$\dim H_n \geq \sum_\lambda f_\lambda^2.$$

But, the term on the right is purely combinatorial, and hence is equal to the dimension of $\mathbb{C}[W_n]$, since the irreducibles for that algebra can be constructed in exactly the same way by Theorem 7.3. Hence,

$$\dim H_n \geq m^n n!$$

as desired. This shows that $\dim H_n = m^n n!$, which implies that H_n is semisimple, as its radical must then be trivial.

□

The proof of Theorem 9.8 and the basis theorem above also give us the branching rule for H_n. We first have the following obvious corollary.

Corollary 9.10. The subalgebra of H_n generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the Hecke algebra of type A_{n-1} and the subalgebra generated by $T_0, \ldots, T_{n-2}$ is isomorphic to H_{n-2}. In fact, this works over any ring, and not just in the generic case.

Then, the branching rule from representations of H_n to representations of H_{n-1} is given by

Corollary 9.11. As an H_{n-1} representation, we have

$$V_\lambda \cong \bigoplus_\mu V_\mu$$

where the sum is taken over m-paritions μ of $n-1$ that are obtained from λ by removing one box.

We end the section with the following remark that describes a sufficient condition for specializations of H_n to be semisimple. If we look at the construction of the irreducibles given above, we see that as long as we specialize at values of q, q_i such that for every d from $-n$ to n and for every i, j from 1 to m, we have

$$q^d \frac{q_i}{q_j} \neq 1$$

then the matrix given in the definition of the irreducible V_λ will be well-defined for any λ. In this case, the specialization of H_n will be semisimple, and the irreducibles will be given by specializations of the V_λ, with the same branching rule.

10. **Symmetric Structure on** H_n

In this section, we return to working over arbitrary $\mathbb{C}$-algebras R and arbitrary unit values of q_i, q. From this section on, H_n denotes the cyclotomic Hecke algebra over R.
We let X be the Jucys-Murphy basis defined as before. For $\mathbf{c} \in \mathbb{Z}^n$ with $0 \leq c_i \leq m-1$, define $L^{\mathbf{c}} = L_1^{c_1} \cdots L_n^{c_n}$. Then, we have an R-linear map $\tau : H_n \to R$ determined by

$$\tau(L^{\mathbf{c}} T_w) = \begin{cases} 1 & \text{if } \mathbf{c} = 0, w = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We want to show that the R bilinear form ν on H_n defined by $\nu(x, y) = \tau(xy)$ is symmetric, nondegenerate and satisfies $\nu(x, zy) = \nu(xz, y)$ for all $x, y, z \in H_n$. The third property is obvious so we focus only on symmetry and non-degeneracy. To prove these properties, we will first construct an alternative R basis for H_n based on reduced words in W_n and then reformulate τ in that basis.

10.1. **Reduced Words in** W_n**.** We begin with the following definitions.

Definition 10.1. A word in W_n is just a word in the alphabet $S = \{s_0, \ldots, s_{n-1}\}$.
We define the length of a word $s_{i_1} \ldots s_{i_n}$ to be n and we say that a word representing $w \in W_n$ is reduced if it has minimal length.
We say that two words are braid equivalent if one can be transformed into the other by using

only the braid relations in W_n.
We sat that two words are weakly braid equivalent if one can be transformed into the other by using the braid relations and one extra relation

$$s_1 s_0^a s_1 s_0^b = s_0^b s_1 s_0^a s_1$$

for each $a, b \in \mathbb{Z}$.

It can be checked that the last relation holds in W_n but is not a consequence of just the braid relations. We next define specific elements in W_n that will be useful in studying braid equivalent and weak braid equivalence of reduced words.

Definition 10.2. For non-negative integers k, a, we define

$$l_{k,a} = \begin{cases} s_{k-1} \cdots s_1 s_0^a & a > 0 \\ 1 & a = 0 \end{cases}.$$

The following Lemma is now an easy exercise in induction.

Lemma 10.3. (a) For any i from 1 to n and any $a \geq 0$, we have up to braid equivalence

$$s_i l_{k,a} = \begin{cases} l_{k,a} s_i & i > k \text{ or} a = 0 \\ l_{k+1,a} & i = k, a \neq 0 \\ l_{k-1,a} & i = k-1, a \neq 0 \\ l_{k,a} s_{i+1} & i < k-1, a \neq 0 \end{cases}.$$

(b) For $a, b > 0$, we have up to weak braid equivalence

$$l_{k,a} l_{k,b} = \begin{cases} l_{k-1,b} l_{k,a} s_1 & k > 1 \\ l_{1,a+b} & k = 1 \end{cases}.$$

(c) For $a, b, k > 1$ and $m \geq 0$, we have up to weak braid equivalence

$$l_{k+m,a} l_{k,b} = l_{k-1,b} l_{k+m,a} s_1.$$

It turns out that unlike in S_n, two reduced words for the same element of W_n are not braid equivalent. However, they are weakly braid equivalent. We prove this in the following theorem, carefully marking where we use the weak braid relation because the weak relations get deformed on passing to the cyclotomic Hecke algebra.

Theorem 10.4. Let

$$x = x_1 \cdots x_k, \ x_i \in S$$

be a reduced word for $x \in W_n$. Then, x is braid equivalent to an expression of the form

$$x = l_{k_1,a_1} \cdots l_{k_r,a_r} w, \ w \in S_n, a_i > 0.$$

and x is weakly braid equivalent to an expression of the form

$$l_{1,a_1'} \cdots l_{n,a_n'} w', \ w' \in S_n, a_i' \geq 0$$

with $\sum_j a_j' = \sum_i a_i$.

Proof. Note that it suffices to prove the first claim, as the consequence is immediate from (c) in Lemma 10.3. We prove the statement by inducting on the length of x, with the base case being trivial. Suppose the reduced expression for x is of the form

$$x = w_1 s_0^{a_1} \cdots w_l s_0^{a_l} w_{l+1}$$

for w_i a reduced word in S_n. Then, by using the braid relations in S_n, we can assume

$$w_1 = s_{k_1-1} \cdot s_1 w_1'$$

with $w_1' \in \langle s_2, \dots, s_{n-1} \rangle$. Since w_1' commutes with s_0, we have under braid equivalence

$$x \equiv l_{k_1,a_1} x'.$$

Since $l(x') < l(x)$ (as $a_1 > 0$), we are done by induction.

□

Corollary 10.5. The second set of expressions in the above theorem give us a complete set of reduced expressions for W_n, with each representative corresponding to a different element.

We now connect reduced words in W_n with reduced words in $H_{A,n}$.

Definition 10.6. For a reduced expression $x = x_1 \cdots x_k$ of $x \in W_n$, define $T_k = T_{x_1} \cdots T_{x_k}$.

Additionally, in analogy with the elements $l_{k,a}$ in W_n, define the elements

$$L_{k,a} = \begin{cases} T_{k-1} \cdots T_1 T_0^a & a > 0 \\ 1 & a = 0 \end{cases}.$$

In $H_{A,n}$, the braid relations still hold but the weak braid relations, and hence the relations in Lemma 10.3 are q-deformed as follows. Again, the proof is left as an exercise and is very similar to the proof in the undeformed case.

Lemma 10.7. (a) For any $a, b \geq 0$, we have

$$T_1 T_0^a T_1 T_0^b = T_0^b T_1 T_0^a T_1 + q(q-1) \sum_{i=1}^{b} T_0^{a+b-i} T_1 T_0^i - T_0^i T_1 T_0^{a+b-i}.$$

(b) For any i from 1 to $n-1$ and $1 \leq k \leq n$,

$$T_i L_{k,a} = \begin{cases} L_{k,a} T_i & i > k, a = 0 \\ L_{k+1,a} & i = k, a \neq 0 \\ q L_{k-1,a} + (q-1) L_{k,a} & i = k-1, a \neq 0 \\ Lk, a T_{i+1} & i < k-1, a \neq 0. \end{cases}$$

(c) For $a, b, m > 0, k > 1$, we have

$$L_{k+m,a} L_{k,b} = L_{k-m-1,b} L_{k,a} T_1 + (q-1) \sum_{i=1}^{b} L_{k-1,a+b-i} L_{k+m,i} - L_{k-1,i} L_{k+m,a+b-i}.$$

The exact expression is unimportant. What is useful is that the sums of the second index remains the same for each monomial term.

Here's the point of all these annoying calculations. Our goal here is to define a linear form on H_n as the coefficient of T_e when a complete set of reduced expressions is used as a basis for H_n. There are, however, two obstructions for this form to be well defined.

(1) If x, x' are two reduced expressions for the same word, then we need $T_x - T_{x'}$ to have 0 T_e coefficient.
(2) For any complete representative set Δ of reduced expressions of elements of W_n, we need the set

$$X_\Delta := \{T_x : x \in \Delta\}$$

to give us an R-basis for H_n.

We prove both these facts as corollaries of the following theorem.

Theorem 10.8. Let x, x' be reduced expressions for the same element in W_n. Then,

$$T_x - T_{x'} \in \sum_{y \notin S_n,\ 0<l(y)<l(x)} AT_y$$

i.e. their difference involves expressions of smaller length that contain nonzero number of T_0 terms.

Proof. If x is a reduced expression involving only the S_n generators, then x' must also only involve the S_n generators and hence this follows from Matsumoto's lemma. So, we assume $x \notin S_n$. Now, since the braid relations still hold in H_n, by the proof of Theorem 10.4, we can write

$$T_x = L_{k_1,a_1} \cdots L_{k_r,a_r} T_w,\ w \in S_n, a_i > 0$$

with $r > 0$ i.e. that there is a T_0 term. Additionally, without loss of generality, we can assume that

$$T_{x'} = L_{1,a'_1} \cdots L_{n,a'_n} T_{w'}$$

with $\sum_i a'_i = \sum_i a_i$. Now, to move T_x to the normalized $T_{x'}$, we repeatedly use the relations in Lemma 10.7 (c). But for any terms that appear in that Lemma, apart from $T_{x'}$, we have

(1) Smaller length.
(2) Same total second L-index.

The second condition implies that since $\sum_{a_i} > 0$ to begin with, any term that appears in

$$T_x - T_{x'}$$

must involve a T_0 term somewhere. Thus, none of these terms lie in T_{S_n}. Hence, we have the desired result

$$T_x - T_{x'} \in \sum_{y \notin S_n,\ 0<l(y)<l(x)} AT_y.$$

□

Corollary 10.9. Let Δ be a complete set of representatives of reduced expressions for elements of W_n. Then, $X_\Delta := \{T_x : x \in \Delta\}$ is an R-basis for H_n.

Proof. By dimension considerations, it suffices to prove that X_Δ spans $H_{A,n}$ over A. We prove by induction that AX_Δ contains T_y for all words (not necessarily reduced) y of length n. For $n = 0$, this is obvious, as X_Δ must contain T_e. Suppose it holds for all words of length $n - 1$ or less and let y be a word of length n.

Now, if y is not reduced, then using the braid and eigenvalue relations we can write T_y as a sum of $T_{y'}$'s with y' all of smaller length. Thus, we can assume that y is reduced. For reduced y, if $y \in S_n$, then T_y is in X_Δ, as all reduced word representatives give the same element in H_n (Matsumoto's Lemma). If T_y is not in S_n, pick some $T_x \in X_\Delta$ with x reduced corresponding to y and then use the previous theorem.

□

Corollary 10.10. Let Δ now be an arbitrary system of representatives and let X_Δ be as before. Note that $T_e \in X_\Delta$ necessarily. Then, the linear form $\tau' : H_{A,n} \to A$ determined by

$$\tau'(T_x) = \begin{cases} 1 & x = e \\ 0 & \text{otherwise} \end{cases}$$

is independent of the choice of Δ.

Our goal now is to show that τ' is symmetric, non-degenerate and that $\tau' = \tau$.. We prove non-degeneracy only in the Hecke algebra defined over $A = R[q^\pm, q_i^\pm]$. Non-degeneracy holds for arbitrary specializations of the cyclotomic Hecke algebra but the proof is purely technical and can be looked up in [MM98].

Theorem 10.11. The bilinear form $\sigma'(x, y) = \tau'(xy)$ from $H_{A,n} \times H_{A,n} \to A$ is non-degenerate.

Proof. It suffices to show that σ' is non-degenerate after specializing at particular values of q, q_i (by discriminant considerations). Pick $q = 1$ and $q_i = \zeta^i$, where ζ is a primitive mth root of unity. Under this specialization, $H_{R,n}$ becomes the group algebra $R[W_n]$ (we assume R is a field by taking its field of fractions if necessary) and the form τ' becomes the standard form on the group algebra that picks out the coefficient of the identity. This form is obviously non-degenerate. Hence, τ' is non-degenerate over A. □

Symmetry requires a little more work.

Theorem 10.12. σ' is a symmetric form.

Proof. Since σ' is independent of the choice of Δ, we choose Δ consisting of reduced words in the normalized form described in the second part of Theorem 10.4. We need to show that for every $x \in \Delta$ and every word y,

$$\tau'(T_x T_y) = \tau'(T_y T_x).$$

By induction on the length of y, we can assume $T_y = T_i$. By induction on the length of x, we can assume it holds for all $T_{x'}$ of smaller length. We now have two cases.

Case $i \geq 1$: Write T_x as

$$T_x = L_{1,a_1} \cdots L_{n,a_n} T_w$$

for $w \in Sn$. Then, it is clear that

$$\tau'(T_xT_i) \neq 0 \Leftrightarrow a_j = 0 \; \forall j, w = s_i.$$

In the case where it is nonzero, $T_x = T_i$ and hence

$$\tau'(T_xT_i) = \tau'(T_iT_x) = \tau'(T_i^2).$$

So, suppose now that $T_x \neq T_i$. Then, by a similar proof to the proof of Theorem 10.4, the word x is also weakly braid equivalent to a reduced word of the form

$$x' = w' l_{n,a'_n} \cdots l_{1,a'_1}.$$

Hence, by Theorem 10.8

$$T_x = T_{x'} + T$$

where T is in the span of elements in X_Δ of smaller length that do not correspond to words in S_n and $x' \neq s_i$ as $x' \notin S_n$. Thus, since $\tau'(TT_i) = 0$ by the above argument, and $\tau'(T_iT_{x'}) = 0$ by a similar argument, by the induction hypothesis, we have

$$\tau'(T_xT_i) - \tau'(T_iT_x) = \tau'(T_xT_i) - \tau'(T_iT_{x'}) + \tau'(T_iT) = \tau'(TT_i) = 0.$$

This finishes case 1.

Case $i = 0$: Again, write

$$T_x = L_{1,a1} \cdots L_{n,a_n} T_w.$$

Then, $T_0T_x \in X_\Delta$ unless $a_1 = m - 1$. This gives us

$$\tau'(T_0T_x) \neq 0 \Leftrightarrow T_x = T_0^{m-1}$$

and in the nonzero case, there is obvious symmetry. So now, suppose $T_x \neq T_0^{m-1}$. We want to show that $\tau'(T_xT_0) = 0$.

In S_n, w is braid equivalent to $s_k \cdots s_1 w'$ for some w' that commutes with s_0. Hence, since the braid relations still hold in H_n

$$T_xT_0 = L_{1,a_0} \cdots L_{n,a_n} T_k \cdots T_1 T_0 T_{w'}$$

which has trace $\tau' = 0$ if $T_{w'} \neq 1$ by the induction hypothesis. So, assume $T_{w'} = 1$. Then,

$$T_xT_0 = L_{1,a_0} \cdots L_{n,a_n} L_{k,1}$$

and using Lemma 10.7 (c) again, we see that this gives us a nonzero T_1 coefficient if and only if $T_x = T_0^{m-1}$.

□

We finish this section by proving that $\tau = \tau'$. This follows immediately from the following Lemma.

Lemma 10.13. Let $i \geq 1$ and let L_i be the Jucys-Murphy element defined in the previous sections. Then,

(a) For $1 \leq k \leq m-1$, L_i^k is an A-linear combination of

$$L_{1,c_1} \cdots L_{i,c_i} T_w$$

with $0 \leq c_j \leq m-1$ for $j \neq i$ and $1 \leq c_i \leq k$.

(b) Let $1 \leq b_i \leq m-1$. Then, $L_1^{b_1} \cdots L_i^{b_i}$ is an A-linear combination of

$$L_{1,c_1} \cdots L_{i,c_i} T_w$$

with $c_i \neq 0$.

Proof. First prove (a) and then (b) using induction on i and Lemma 10.7 (c). □

This lemma tells us that T_e coefficients in the word basis and the Jucys-Murphy basis agree. Hence, $\tau = \tau'$.

Part C - Affine Hecke Algebra Modules in the Kato Block

This is the last section of these notes in which we construct an equivalence of categories between representations of $\mathcal{H}_{F,n}$ of particular central character and representations of $Z_{F,n}$ of particular formal character, for which the essential tool will be the unique irreducibility of a Kato Module in its block. In this section, we fix F as an algebraically closed field (not necessarily characteristic 0) and then omit the F from the notation for the various algebras.

Additionally, in this section, we fix an $a \in F^\times$ and restrict ourselves to $\mathcal{H}_n$-modules that are in the block with central character corresponding to $(a, \ldots, a)$ i.e. in the block in which the unique irreducible is the Kato Module $L(a^n)$, which we now denote as K_n. We thus make the following definitions in order to formulate the results better.

Definition 10.14.

1. Let $\mathfrak{m}_n$ be the maximal ideal in Z_n that is the intersection of the two sided ideal of $\mathcal{P}_n$ generated by $(X_1 - a, \cdots X_n - a)$. Let $\widehat{Z_n}$ denote the completion of Z_n at this maximal ideal and let $\widehat{\mathcal{P}_n}, \widehat{\mathcal{H}_n}$ be the completion of the respective algebras at $\mathfrak{m}_n$. Let $\widehat{\mathcal{H}_n^T}$ be the completion of $\mathcal{H}_n^T$ inside $\widehat{\mathcal{H}_n}$.

2. Define $\mathcal{M}_n$ to be the category of Z_n-modules on which $\mathfrak{m}_n$ acts locally nilpotently, or equivalently as the category of $\widehat{Z_n}$-modules. Similarly, define $\mathcal{N}_n$ to be the category of left $\mathcal{H}_n$-modules on which $\mathfrak{m}_n$ acts locally nilpotently, or equivalently as the category of $\widehat{\mathcal{H}_n}$-modules.

Our goal is to prove that $\mathcal{M}_n$ and $\mathcal{N}_n$ are equivalent and, moreover, to specify a pair of functors that establish the equivalence of categories. Before we can state this precisely, we need to define the following element in $\mathcal{H}_n$:

Definition 10.15. Let τ be either the trivial or sign character on H_n^T. Then, define the elements $c_n^\tau \in Z(H_n^T)$ as

$$c_n^{\mathrm{sgn}} := \sum_{w \in S_n} q^{-l(w)} \tau(T_w) T_w$$

and

$$c_n^1 := \sum_{w \in S_n} T_w.$$

For each τ, c_n^τ has some nice properties.

Proposition 10.16. The following hold:

1. For any $T_w \in \mathcal{H}_n^T$, we have
$$T_w c_n^\tau = c_n^\tau T_w = \tau(w) c_n^\tau.$$
and in particular, for $n > 1$,
$$c_n^1 c_n^{\mathrm{sgn}} = 0.$$
2. For any projective $\mathcal{H}_n^T$-module M,
$$c_n^\tau(M) = \{m \in M : T_w(m) = \tau(w) m \ \forall w \in S_n\}.$$
3. For any projective $\mathcal{H}_n^T$-module M, the multiplication map
$$c_n^\tau \mathcal{H}_n^T \otimes_{\mathcal{H}_n^T} M \to c_n^\tau M$$
is an isomorphism.
4. For any $\mathcal{H}_n$-module M, the canonical map
$$c_n^\tau \mathcal{H}_n^T \otimes_{\mathcal{H}_n^T} M \to c_n^\tau \mathcal{H}_n \otimes \mathcal{H}_n M$$
is an isomorphism of Z_n-modules.

Proof. 1 is direct computation. 2 follows by looking at the case where M is free which follows from 1. 3 is straightforward from 2 and 4 follows from 3. □

Remark. Note that in characteristic 0, the above proposition implies that the two c_n^τ are mutually orthogonal simple central idempotents and in this case, the entire results of this section follow from the theory of central idempotents. In the case of positive characteristic, however, for large values of n, c_n^τ will actually be nilpotent and hence the results proved below are nontrivial.

We are now ready to state and prove the main theorem.

Theorem 10.17. Fix some $\tau \in \{1, \mathrm{sgn}\}$. Define the functor $\mathcal{F} : \mathcal{M}_n \to \mathcal{N}_n$ as

$$M \mapsto \widehat{\mathcal{H}_n} c_n^\tau \otimes_{\widehat{Z_n}} M$$

and the functor $\mathcal{G} : \mathcal{N}_n \to \mathcal{M}_n$ as

$$M \mapsto c_n^\tau \widehat{\mathcal{H}_n} \otimes_{\widehat{\mathcal{H}_n}} M.$$

Then, F and G establish an equivalence of categories between $\mathcal{M}_n$ and $\mathcal{N}_n$.

Proof. We break the proof down into several steps.

Step 1: We show that $\mathcal{F}$ and $\mathcal{G}$ are exact functors.
$\mathcal{F}$ is exact because, by 1 in Proposition 10.16, $\widehat{\mathcal{H}_n}c_n^\tau$ is free of rank 1 over $\mathcal{P}_n$ and is hence free of rank $n!$ over Z_n and hence is flat over Z_n.
Now, $\mathcal{G}$ is clearly right exact. To show that $\mathcal{G}$ is left exact, note that every module in $\mathcal{N}_n$ has a filtration by the Kato module K_n, as it is the only irreducible in the category. But K_n is free over $\mathcal{H}_n^T$ and hence every module in $\mathcal{N}_n$ is projective over $\mathcal{H}_n^T$. Thus, by Proposition 10.16, the functor $\mathcal{G}$ is isomorphic to the functor

$$M \mapsto M^\tau := \{m \in M : h \cdot m = \tau(h)m \ \forall h \in \mathcal{H}_n^T\}.$$

This functor is clearly left exact.

Step 2: We show that $\mathcal{G}$ is right adjoint to $\mathcal{F}$. Note that $\mathcal{F}$ has an obvious right adjoint $\mathcal{F}^* : \mathcal{N}_n \to \mathcal{M}_n$ which sends

$$M \mapsto \mathrm{Hom}_{\widehat{H_n}}(\widehat{H_n}c_n^\tau, M).$$

Now, $\widehat{H_n}c_n^\tau$ is isomorphic as a left module to $\widehat{H_n}/I$, where by Proposition 10.16 and the Basis Theorem, I is the ideal $\widehat{P_n} \otimes I^T$ with I^T the left ideal in $\widehat{\mathcal{H}_n^T}$ generated by $(h - \tau(h))$. Then, by the proof of left exactness of G, we have canonical isomorphisms

$$\mathcal{F}^*(M) \cong M^I := \{m \in M : Im = 0\} = M^\tau \cong \mathcal{G}(M).$$

Hence, $\mathcal{G}$ is right adjoint to $\mathcal{F}$.
Step 3: We finish the proof of the theorem by showing that the counit and unit are isomorphisms. Since the functors $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are exact it suffices to show this fact for the simple objects. In $\mathcal{N}_n$, we have a unique simple K_n and in $\mathcal{M}_n$, we have the unique one dimensional simple corresponding to the central character $(a, \cdots, a)$, which we denote now by L_n.

Now, as $\widehat{\mathcal{H}^n}c_n^\tau = \widehat{\mathcal{P}_n}c_n^\tau$ is free of rank $n!$ over $\widehat{\mathbb{Z}_n}$,

$$\dim_F \mathcal{F}(L_n) = n! \Rightarrow \mathcal{F}(L_n) \cong K_n$$

as $\mathcal{F}(L_n)$ must contain the Kato module and both have the same dimension. Conversely, since the action of $\widehat{\mathcal{H}_n^T}$ on K_n is just the action of $\widehat{\mathcal{H}_n^T}$ on itself, we see that

$$\dim_F \mathcal{G}(K_n) = \dim_F K_n^\tau = 1 \Rightarrow G(K_n) \cong L_n.$$

Thus, the unit and counit are both nonzero morphisms between the same simple objects and are hence isomorphisms.

□

Remark. There is an alternative approach to proving the main theorem. We state this approach here but do not carry it through. To show that $\mathcal{F}$ and $\mathcal{G}$ establish an equivalence of categories, it suffices to prove that

$$c_n^\tau \widehat{\mathcal{H}_n} \otimes_{\widehat{\mathcal{H}_n}} \widehat{\mathcal{H}_n} c_n^\tau \cong \widehat{\mathbb{Z}_n}$$

as a left $\widehat{\mathbb{Z}_n}$ module and

$$\widehat{\mathcal{H}_n} c_n^\tau \otimes_{\widehat{\mathcal{Z}_n}} c_n^\tau \widehat{\mathcal{H}_n} \cong \widehat{\mathcal{H}_n}$$

as a left $\widehat{\mathcal{H}_n}$ module. This is the approach taken in [CR04].

Appendix

Our construction of the irreducible representations of H_n is via a deformation of the Specht module construction for W_n. We first define standard Young tableaux, content and axial distances for m-partitions of n.

Definition 10.18. Let λ be an m-partition of n. Then, a standard Young tableau for the l-tableau associated to λ is a filling in of the boxes of the m-tableau with the numbers $1, \ldots, n$ such that in each component tableau, the enumeration is standard.

Let a, b now be boxes in λ. Then, we define the content $c(\lambda; a)$ of a to be the row index of a minus the column index of a, in the component tableau in which a lives. Additionally, we define the axial distance

$$r(a, b) = c(\lambda, a) - c(\lambda, b).$$

Finally, for an integer l and an indeterminate y, we first define

$$\Delta(l, y) = 1 - q^l y \in A[y]$$

and then define the following 2×2 matrix

$$M(l, y) = \frac{1}{\Delta(l, y)} \begin{pmatrix} q - 1 & \Delta(l+1, y) \\ q\Delta(l-1, y) & -q^l y(q-1) \end{pmatrix}.$$

Now, let $\lambda \vdash_m n$ and let V_λ be the free K-vector space with basis the set of standard Young tableaux of shape λ. For any $i \in \{1, \ldots, n\}$ and for any standard Young tableau M_ρ of shape λ, define $\tau_\lambda(i)$ to be the index of the component tableau of ρ that i appears in. We now define a representation of H_n on V_λ as follows:

(1) $T_0 M_\rho = q_{\tau_\rho(i)} t_\rho$.

(2) For $i > 0$, we have 3 cases for the action of T_i on M_ρ:
   a. If $i, i+1$ lie in the same row of the same component diagram of M_ρ, then $T_i M_\rho = q M_\rho$.

   b. If $i, i+1$ lie in the same column of the same component diagram of M_ρ, then $T_i M_\rho = -1 M_\rho$.

   c. If neither of the above hold, let $M_{\rho'}$ be the standard Young tableau with i and $i+1$ swapped. Then,

$$T_i\langle M_\rho, M_{\rho'}\rangle = \langle M_\rho, M_{\rho'}\rangle M\left(r(i+1,i), \frac{q_{\tau_\rho(i)}}{q_{\tau_\rho(i+1)}}\right).$$

This is well-defined because if i, $i+1$ lie in the same component diagram, then the axial distance between them cannot be $0, 1, -1$ unless they are in the same row or column.

As an example, and also because it will be useful in proof of irreducibility , we compute the matrix corresponding to the action of T_{n-1} in case 2(c). We let d denote $r(n, n-1)$, a denote $\tau_\rho(n-1)$ and b denote $\tau_\rho(n)$. Then, the action of T_{n-1} is given by the matrix

$$M\left(r(n,n-1), \frac{q_{\tau_\rho(n-1)}}{q_{\tau_\rho(n)}}\right) = \frac{1}{1-q^d\frac{q_a}{q_b}}\begin{pmatrix} q-1 & \frac{1}{1-q^{d+1}\frac{q_a}{q_b}} \\ \frac{q}{1-q^{d-1}\frac{q_a}{q_b}} & -q^d(q-1)\end{pmatrix}.$$

We denote this special matrix by N, and note that, since q is a unit, N_{21} is nonzero. The proof of irreducibility now follows from what was discussed in the main section.

References

[AK94] S Ariki and K Koike. A hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations. *Adv. Math*, 106:216–293, (1994).

[BM97] K. Bremke and G. Malle. Reduced words and a length function for $G(e, 1, n)$. *Indag. Math. (N.S.)*, 8:453–469, (1997).

[CR04] J. Chuang and R. Rouquier. Derived equivalencs for symmetric groups and $\mathfrak{sl}_2$categorification. *ArXiv Mathematics e-prints*, jul 2004.

[GJ11] Meinolf Geck and Nicolas Jacon. *Representations of Hecke Algebras at Roots of Unity*, volume 15 of *Algebra and Applications*. Springer-Verlag, 2011.

[Kle05] Alexander Kleshchev. *Linear and Projective Representations of Symmetric Groups*. Cambridge Tracts in Mathematics. Cambridge University Press, 2005.

[Mac95] I. G. Macdonald. *Symmetric Gunctions and Hall Polynomials, Second Edition*. Oxford Math. Monographs. Clarendron Press Oxford, 1995.

[MM98] G. Malle and A. Mathas. Symmetric cyclotomic hecke algebras. *J. Algebra*, 205:275–293, (1998).