THE HODGE THEORY OF SOERGEL BIMODULES

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Abstract. These are notes for a talk at the MIT-Northeastern Fall 2017 seminar on category $\mathcal{O}$ and Soergel bimodules. This talk covers the first three sections of the paper [EW] of Elias-Williamson with the same title and is entirely based on it. These notes contain no original material.

Contents

1. Introduction

In the first section we'll give an outline of the structure of the Elias-Williamson proof of Soergel's conjecture appearing in [EW]. Soergel's conjecture makes some statement $S(x)$ for every element x of a Coxeter group W. In particular, it lends itself to an inductive proof. However, the inductive proof is quite complicated and involves several auxiliary statements, again indexed by elements or subsets of W, having to do with certain invariant forms on Soergel bimodules. If you've seen Kashiwara's grand loop argument for the existence of crystal bases at some point, it's of a similar flavor (the inductive structure, not the mathematical content). We will view the content of the paper [EW] as the material to be discussed, taking (as [EW] does) Soergel's previous work (e.g. [S2]) for granted. OK, let's get to it:

2. Navigating the Proof

In order to motivate discussing "Lefschetz linear algebra", "Hodge-Riemann bilinear relations", etc., it is useful just to have a roadmap of where we're going. This section provides that roadmap and can be viewed as an extended abstract. In order to make certain statements I'll need to be at least halfway precise, and as I'll need to be fully precise later I will just be fully precise now and include all the new definitions and statements.

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2.1. **Soergel's Conjecture vs. the Kazhdan-Lusztig Positivity Conjecture.** Throughout this talk, fix a Coxeter system (W, S). Crucially, we will not assume that W is a Weyl group, or even finite. Recall that attached to (W, S) we have the *Hecke algebra* $\mathcal{H}$, a $\mathbb{Z}[v^{\pm 1}]$-algebra defined in previous lectures. Recall in particular that there are(at least) two natural bases for $\mathcal{H}$: the *standard* basis $\{H_x\}_{x \in W}$ and the Kazhdan-Lusztig basis $\{\underline{H}_x\}_{x \in W}$ [KL1]. The standard basis is elementary, while the Kazhdan-Lusztig basis is in general quite complicated. In fact, it is complicated enough that the following statement about the transition matrix and structure constants for the multiplication went unanswered for general W for decades:

Conjecture 2.1.1 (Kazhdan-Lusztig Positivity Conjecture [KL1])**.** *For any Coxeter system* (W, S)*:*

(1) If we write $\underline{H}_x = \sum_{y \leq x} h_{y,x} H_y$ *then* $h_{y,x} \in \mathbb{Z}_{\geq 0}[v]$*,*
(2) If we write $\underline{H}_x \underline{H}_y = \sum \mu_{x,y}^z \underline{H}_z$ *then* $\mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$*.*

A common approach to prove such positivity statements, and the approach we are discussing in this seminar, is to identify the quantities that are supposed to be positive with quantities that are manifestly positive: dimensions of vector spaces, multiplicities in more general algebraic contexts, enumeration of some kind of objects, etc. This is typically achieved by producing an "enriched" version of the problem at hand, recast in geometric or category-theoretic terms (a *categorification*). For an elementary example, consider the algebra of symmetric functions and its basis of Schur functions. That the structure constants for the multiplication are positive is obvious once one realizes that the algebra of symmetric functions is categorified by representations of the symmetric groups (Schur functions correspond to the irreducible representations, and the multiplication corresponds to the induction product).

In the case that W is a finite or affine Weyl group, one has the entire kitchen sink of geometric representation theory to throw at the problem. In particular, shortly after making their positivity conjecture, Kazhdan and Lusztig proved it [KL2] in the case that W is a finite or affine Weyl group by relating the polynomials $h_{y,x}$ to the Poincaré polynomials of the intersection cohomology of Schubert varieties. However, it is unclear how to generalize this approach to general W: one doesn't have the Schubert varieties to work with in that case!

A decade later, Soergel [S1] gave an alternative proof for Weyl groups, nearly entirely algebraic in nature except at one point again using the (equivariant) intersection cohomology of Schubert varieties. It is this algebraic framework (*Soergel bimodules*) that Elias and Williamson used to prove the Kazhdan-Lusztig positivity conjecture for general W; the main (and very nontrivial!) hurdle is to find a way to circumvent the geometric arguments used by Soergel.

In fact, Elias-Williamson proved a stronger conjecture (whatever it means for one true statement to be stronger than another), *Soergel's conjecture*, that implies the Kazhdan-Lusztig positivity conjecture. In particular, recall the category $\mathcal{B}$ of Soergel bimodules associated to the Coxeter system (W, S) and real reflection representation $\mathfrak{h}$ (one needs to take care with the *choice* of reflection representation, it should be "reflection faithful" - a bit more on this later). The category $\mathcal{B}$ is the full additive monoidal Karoubian subcategory of graded R-bimodules generated by the bimodules $B_s := R \otimes_{R^s} R(1)$ for all $s \in S$, where $R = \mathbb{R}[\mathfrak{h}]$ is the algebra of polynomial functions on $\mathfrak{h}$ graded with $\deg \mathfrak{h}^* = 2$. In particular, we may consider its split Grothendieck group $[\mathcal{B}]$. As $\mathcal{B}$ is also monoidal and its objects are graded, $[\mathcal{B}]$

is a $\mathbb{Z}[v^{\pm 1}]$ algebra, with the product reflecting the monoidal structure $[B_1][B_2] := [B_1 \otimes B_2]$ and the grading reflected in $v[B] := [B(1)]$. Typical objects in $\mathcal{B}$ are the *Bott-Samelson* bimodules $BS(\underline{x})$ attached to any expression (= finite sequence of simple reflections) $\underline{x} = (s_1, ..., s_n)$:

$$BS(\underline{x}) := B_{s_1} \cdots B_{s_n},$$

where juxtaposition denotes the tensor product over R.

Soergel's *categorification theorem* then makes several assertions (see Boris' talk for a detailed discussion). First, for any $x \in W$ and any reduced expression $\underline{x}$ for x, there is a unique (up to isomorphism) indecomposable bimodule B_x appearing as a direct summand in $BS(\underline{x})$ but not appearing as a shifted direct summand of any $BS(\underline{y})$ for a shorter expression $\underline{y}$, and B_x only depends on the element $x \in W$, and not on the choice of reduced expression $\underline{x}$, up to isomorphism. Furthermore, the bimodules B_x for $x \in W$ give a nonredundant set of representatives of the isomorphism classes of indecomposable objects in $\mathcal{B}$ up to shift. The bimodule B_x is cyclic and generated in degree $-l(x)$. Furthermore, $[\mathcal{B}]$ is isomorphic to the Hecke algebra $\mathcal{H}$ via the assignments $\underline{H}_s \mapsto [B_s]$ for $s \in S$. The inverse isomorphism, $\mathrm{ch} : [\mathcal{B}] \to \mathcal{H}$, sends a Soergel bimodule B to a $\mathbb{Z}_{\geq 0}[v^{\pm}]$-linear combination of the *standard* basis elements H_y, where the coefficient of H_y records the graded multiplicity of certain *standard bimodules* (which are *not* typically Soergel bimodules!) in a certain canonical filtration. In particular, for every $x \in W$ we can define the R-bimodule R_x which is the free rank 1 R left R-module R with right R-module structure twisted by x, i.e. $m.r = x(r)m$ for all $m \in R_x$ and $r \in R$. This is just the R-bimodule structure on the regular functions on the linear subspace ("twisted graph")

$$Gr(x) = \{(xv, v) | v \in \mathfrak{h}\} = \{(v, x^{-1}v) | v \in \mathfrak{h}\} \subset \mathfrak{h} \times \mathfrak{h}.$$

For any subset $A \subset W$ we can consider the subspace

$$Gr(A) = \bigcup_{x \in A} Gr(x).$$

For any R-bimodule B we can then define the submodule

$$\Gamma_A B := \{m \in B | \mathrm{Supp}(m) \subset Gr(A)\},$$

and in particular for any $x \in W$ we define $\Gamma_{\geq x} B$ and $\Gamma_{>x} B$ in the obvious way. It's a theorem of Soergel that for any Soergel bimodule B there is a finite subset $A \subset W$ such that $B = \Gamma_A(B)$ and that for all $x \in W$ we have an isomorphism

$$\Gamma_{\geq x} B / \Gamma_{>x} B \cong R_x(-l(x))^{\oplus h_x(B)}$$

for some polynomial $h_x(B) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. The character $\mathrm{ch}(B) \in \mathcal{H}$ is then defined by

$$\mathrm{ch}(B) = \sum_{x \in W} h_x(B) H_x.$$

Conjecture 2.1.2 (Soergel's Conjecture)**.** *For all $x \in W$ we have $ch(B_x) = \underline{H}_x$.*

The Kazhdan-Lusztig positivity conjecture clearly follows immediately from Soergel's conjecture, and it is the latter proved in [EW]. The Kazhdan-Lusztig conjecture about the BGG category $\mathcal{O}$ follows as well.

2.2. **Some Background.** For a Soergel bimodule B, let $\overline{B} := B \otimes_R \mathbb{R}$, where $\mathbb{R} = R/R^+$ and R^+ is the (graded) ideal of polynomials on $\mathfrak{h}$ vanishing at 0. Clearly, $\overline{B}$ retains the structure of a graded left R-module. Let $(\overline{B})^i$ denote its degree i component. Notice that any B_s for simple s is isomorphic to $R(1) \oplus R(-1)$ as a right and left R-bimodule. It follows that $\dim_{\mathbb{R}} \overline{BS(\underline{w})} = 2^{l(\underline{w})}$, its graded dimension is given by binomial coefficients, and it is zero in degrees d with $|d| > l(\underline{w})$.

Throughout, we'll fix the reflection representation $\mathfrak{h}$ and linearly independent elements $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ ("simple roots") and $\{\alpha_s^\vee\}_{s\in S} \in \mathfrak{h}$ ("simple coroots") such that

$$s(\alpha_s) = -\alpha_s, \quad s(\alpha_s^\vee) = -\alpha_s^\vee, \quad \text{and} \quad \alpha_s(\alpha_t^\vee) = -2\cos(\pi/m_{st})$$

where m_{st} is the order of st in W (and $\pi/\infty := 0$). We'll assume that $\mathfrak{h}$ is minimal with these properties. (Technical point: such a reflection representation exists, is *reflection faithful* as discussed above, the results of Soergel [S2] apply, and by results of Libedinsky [Li] it suffices to prove Soergel's conjecture for such an $\mathfrak{h}$.) The action of W on $\mathfrak{h}$ is given by

$$s \cdot v = v - \alpha_s(v)\alpha_s^\vee.$$

That the coroots $\alpha_s^\vee$ are linearly independent means that the corresponding map $\mathbb{R}^{|S|} \to \mathfrak{h}$ is injective. Taking duals, the corresponding evaluation map $\mathfrak{h}^* \to \mathbb{R}^{|S|}$ is surjective; considering the preimage of $(\mathbb{R}^{>0})^{|S|}$, we see that the intersection of half spaces

$$\bigcap_{s\in S} \{v \in \mathfrak{h}^* : v(\alpha_s) > 0\}$$

is nonempty. We'll fix an element $\rho \in \mathfrak{h}^*$ in this intersection of half spaces throughout.

Definition 2.2.1. *A graded R-valued form $\langle -, - \rangle : B \times B \to R$ on a graded R-bimodule B is called* invariant *if it satisfies*
(1) $\langle rb, b' \rangle = \langle b, rb' \rangle$
(2) $\langle br, b' \rangle = \langle b, b'r \rangle = \langle b, b' \rangle r$.

Notice the right-left asymmetry - this asymmetry corresponds to the arbitrary decision to quotient by R^+ on the right rather than on the left. That the form is graded is the statement that $\deg\langle b, b' \rangle = \deg(b) + \deg(b')$.

Definition 2.2.2. *Let M, N be graded R-bimodules. Define the graded hom by*

$$Hom^\bullet(M, N) := \bigoplus_{i\in\mathbb{Z}} Hom(M, N(i)).$$

Define the graded hom $Hom^\bullet_{-R}(M, N)$ similarly. Define the dual *bimodule $\mathbb{D}M$ of M by*

$$\mathbb{D}M := Hom^\bullet_{-R}(M, R),$$

with the structure of an R-bimodule by $(r_1 f r_2)(b) = f(r_1 f r_2)$. Call M self-dual *if $M \cong \mathbb{D}M$.*

Recall the following: ([S2, Satz 6.14]):

Theorem 2.2.3 (Soergel). *Every indecomposable Soergel bimodule B_x is self-dual.*

Obviously, definitions 2.2.1 and 2.2.2 were made for each other. An invariant form on a graded R-bimodule B is the same thing as a graded R-bimodule map

$$B \to \mathbb{D}B,$$

$b \mapsto \langle b, \bullet \rangle$. Such an invariant form is called *non-degenerate* if it induces an isomorphism $B \cong \mathbb{D}B$ (this is stronger than saying the form has a trivial radical). Clearly, an invariant form $\langle -, - \rangle_B$ on B induces a graded $\mathbb{R}$-valued form on $\overline{B}$, denoted $\langle -, - \rangle_{\overline{B}}$, that is invariant for the left R-action.

Corollary 2.2.4. *For all $x \in W$, $ch(B_x)$ is fixed by the bar involution $\overline{v} = v^{-1}$, $\overline{H_x} = H_{x^{-1}}^{-1}$ on $\mathcal{H}$, has coefficient 1 for H_x in the standard basis, and all other standard basis elements H_y appearing with nonzero coefficient satisfy $y < x$.*

Proof. (Included for completeness and to be omitted in the talk itself.) The proof is by induction $l(x)$. Clearly this is true for $l(x) \leq 1$. For general x, assume $\mathrm{ch}(B_y)$ is fixed by bar for all $y \leq x$. Pick a reduced expression $\underline{x}$ for x. Then $\mathrm{ch}(BS(\underline{x}))$ is a product of elements $\underline{H}_s$ for simple reflections x, and so is fixed by bar. Writing $BS(\underline{x}) = B_x \oplus \bigoplus_{y<x} p_y(v) B_y$ for various polynomials $p_y(v) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ by Soergel's theorem, by the self-duality of the B_z we see that the polynomials $p_y(v)$ satisfy $\overline{p_y} = p_y$. Writing $\mathrm{ch}(B_x) = \mathrm{ch}(BS(\underline{x})) - \sum_{y<x} p_y \mathrm{ch}(B_y)$ the claim follows. □

Definition 2.2.5. *If $V = \bigoplus_i V^i$ is a graded finite-dimensional $\mathbb{R}$-vector space, then we define*

$$\underline{\dim} V := \sum (\dim V^i) v^{-i} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}].$$

If M is a finitely generated graded R-module M, set

$$\underline{rk}\ M := \underline{\dim}(M \otimes_R \mathbb{R}).$$

This looks funny (v^{-i} rather than v^i) but is rigged so that $\underline{\dim}(V^{\oplus p}) = p\underline{\dim} V$ for $p \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

For one last detour, let's recall a couple trivial facts about the bar involution and an important bilinear form on the Hecke algebra $\mathcal{H}$. For $p \in \mathbb{Z}[v^{\pm 1}]$, we define $\overline{p}(v) = p(v^{-1})$. This is extended to the bar (ring) involution $\mathcal{H}$, $h \mapsto \overline{h}$ on $\mathcal{H}$ by $\overline{H_x} = H_{x^{-1}}^{-1}$. Recall that the elements $\overline{H}_x$ are fixed by this involution (by definition). Let $(-,-)$ denote the bilinear $\mathbb{Z}[v^{\pm 1}]$-valued form on $\mathcal{H}$ determined by $(H_x, H_y) = \delta_{x,y}$ (standard basis here). A trivial but important property, used below, is that

$$(\underline{H}_x, \underline{H}_y) \in \delta_{xy} + v\mathbb{Z}[v].$$

This form is very important for Soergel bimodules due to the following result of Soergel, which we will take for granted:

Theorem 2.2.6 (Soergel's Hom Formula)**.** *Suppose B, B' are Soergel bimodules. Then $Hom^\bullet(B, B')$ is a graded free right R-module rank*

$$rk\ Hom^\bullet(B, B') = \overline{(ch(B), \overline{ch(B')})}.$$

For example, when Soergel's conjecture holds for x, i.e. when $\mathrm{ch}(B_x) = \underline{H}_x$, it follows from Soergel's hom formula that $\mathrm{Hom}^\bullet(B_x, B_x)$ is concentrated in positive degrees and $\dim \mathrm{Hom}(B_x, B_x) = 1$ (*wakefulness test*: why did we already know this?). In particular:

Lemma 2.2.7. *Suppose $ch(B_x) = \underline{H}_x$ holds. Then B_x admits an invariant form which is unique up to a scalar. Moreover, any nonzero invariant form is non-degenerate.*

Proof. By Theorem 2.2.3, B_x admits a nondegenerate invariant form. The rest follows immediately from Theorem 2.2.6. □

2.3. **Proof Outline.** We now will start defining various statements involved in the inductive proof. The proof will be by induction on the Bruhat order.

Definition 2.3.1. *For $x \in W$, let $S(x)$ be the statement that Soergel's conjecture holds for x. Similarly, if $X \subset W$ is a subset, let $S(X)$ be the statement that $S(x)$ is true for all $x \in X$. Let $S(< x)$ be the statement that $S(y)$ holds for all $y < x$, $S(\leq x)$ is similar, etc.*

Definition 2.3.2. *Let for $x \in W$, let $hL(x)$ ("hard Lefschetz for x") be the statement that for all $i \geq 0$ the operator of left multiplication by ρ^i induces an isomorphism*

$$\rho^i : (\overline{B_x})^{-i} \cong (\overline{B_x})^i.$$

Notations like $hL(X), hL(< x), hL(\leq x)$, etc., are defined similarly.

When $S(x)$ is known, we will fix a nondegenerate invariant form $\langle -, - \rangle_{B_x}$ on B_x, chosen so that the associated form $\langle -, - \rangle_{\overline{B_x}}$ on $\overline{B_x}$ satisfies

$$\langle \overline{c}, \rho^{l(x)} \overline{c} \rangle_{\overline{B_x}} > 0$$

for any generator $c \in B_x^{-l(x)}$. By Lemma 2.2.7, this form is unique up to a positive real multiple. It will be called the *intersection form* on B_x. Note that it is a symmetric form (**asks wakefulness test**).

Definition 2.3.3. *For $x \in W$, let $HR(x)$ ("Hodge-Riemann" for x) be the statement that $S(x)$ holds and that for all $i \geq 0$ the "Lefschetz" form on $\overline{B_x}^{-i}$ defined by*

$$(\alpha, \beta)^{\rho}_{-i} := \langle \alpha, \rho^i \beta \rangle_{\overline{B_x}}$$

is $(-1)^{(-l(x)+i)/2}$-definite when restricted to the primitive subspace

$$P_{\rho}^{-i} := \ker(\rho^{i+1}) \subset (\overline{B_x})^{-i}.$$

The statements $HR(X), HR(< x)$, etc. are defined similarly.

It is annoying that we have to assume $S(x)$ to talk about $HR(x)$. So we'll also introduce related statements $HR(\underline{x})$, where $\underline{x}$ is a reduced expression for x, as follows. We'll see later by an inductive argument/explicit construction that each Bott-Samelson bimodule $BS(\underline{x})$ is equipped with a symmetric non-degenerate *intersection form* coming from the natural ring structure and a certain trace function.

Definition 2.3.4. *For $x \in W$ and $\underline{x}$ a reduced expression for x, let $HR(\underline{x})$ denote the statement that the obvious analogue of $HR(x)$ holds when one replaces the intersection form on B_x (which requires $S(x)$, which we are *not* requiring here!) with the restriction of the intersection form on $BS(\underline{x})$ to B_x (for any embedding of B_x in $BS(\underline{x})$ as a direct summand).*

Lemma 2.3.5. *If $S(x)$ holds, then $HR(x)$ and $HR(\underline{x})$ are equivalent for any reduced expression $\underline{x}$ for x.*

We'll prove this lemma later in Section 3.

ASSUMPTION: Fix $x \in W$ and $s \in S$ with $xs > x$ and assume $S(< xs)$ holds.

Lemma 2.3.6. *$S(< xs)$ implies that $End(B_y) = \mathbb{R}$ for all $y < xs$.*

Proof. Immediate from Soergel's hom formula. □

The following statements are needed for the inductive argument.

Consider the form given by composition

$$(-,-)^{x,s}_y : \mathrm{Hom}(B_y, B_x B_s) \times \mathrm{Hom}(B_x B_s, B_y) \to \mathrm{End}(B_y) = \mathbb{R}.$$

Soergel [S2, Lemma 7.1(2)] showed that $S(xs)$ is equivalent to the non-degeneracy of $(-,-)_y^{x,s}$ for all $y < xs$. As we will see later, B_y and B_xB_s are naturally equipped with nondegenerate symmetric invariant bilinear forms, so there is a canonical identification

$$\mathrm{Hom}(B_y, B_xB_s) = \mathrm{Hom}(B_xB_s, B_y)$$

given by taking adjoints. We will therefore view $(-,-)_y^{x,s}$ as a bilinear form on $\mathrm{Hom}(B_y, B_xB_s)$.

Definition 2.3.7. *$S_\pm(y,x,s)$ ("Soergel's conjecture for (y,x,s) with signs") is the statement that $(-,-)_y^{x,s}$ is $(-1)^{(l(x)+1-l(y))/2}$-definite.*

In particular, by Soergel's result [S2, Lemma 7.1(2)], we have

Lemma 2.3.8. *$S(< xs)$ and $S_\pm(< xs, x, s)$ imply $S(xs)$.*

Definition 2.3.9. *$hL(x,s)$ ("hard Lefschetz for (x,s)") is the statement that for all $i \geq 0$*

$$\rho^i : (\overline{B_xB_s})^{-i} \to (\overline{B_xB_s})^{i}$$

is an isomorphism

Lemma 2.3.10. *$hL(x,s)$ implies $hL(xs)$. If $hL(< xs)$ is known, they are equivalent.*

Proof. The first statement holds because B_{xs} is a summand of B_xB_s. The second statement follows similarly from the standing assumption $S(< xs)$. □

Given a reduced expression $\underline{x}$ for x and an embedding (always as a direct summand) of B_x in $BS(\underline{x})$, then B_x inherits an invariant form from $BS(\underline{x})$. Similarly, B_xB_s is a summand of $BS(\underline{x}s)$ and inherits an invariant form that we'll denote $\langle -,-\rangle_{B_xB_s}$.

Definition 2.3.11. *$HR(\underline{x},s)$ ("Hodge-Riemann for $(\underline{x},s)$") is the statement that for any embedding $B_x \subset BS(\underline{x})$ the Lefschetz form*

$$(\alpha,\beta)_\rho^{-i} := \langle \alpha, \rho^i\beta\rangle_{\overline{B_xB_s}}$$

is $(-1)^{(l(x)+1-i)/2}$-definite on the primitive subspace

$$P_\rho^{-i} := \ker(\rho^{i+1}) \subset (\overline{B_xB_s})^{-i}.$$

Lemma 2.3.12. *$HR(\underline{x},s)$ (defined right above) implies $HR(\underline{x}s)$ (here $\underline{x}s$ denotes an expression).*

Proof. This follows from the fact that if $V^\bullet$ is a finite-dimensional graded vector space with a graded nondegenerate form and Lefschetz operator L satisfying the hard Lefschetz theorem and Hodge-Riemann bilinear relations as in Xiaolei's talk, then any L-stable graded subspace $W \subset V$ with symmetric Betti numbers also satisfies Hodge-Riemann (the restriction of a definite form is definite with the same sign). □

Definition 2.3.13. *$HR(x,s)$ is the statement that $HR(\underline{x},s)$ holds for all reduced expressions $\underline{x}$ for x.*

In a later talk, we will prove the following ([EW, Theorem 4.1]):

Theorem 2.3.14. *There exists an embedding*

$$\iota : Hom(B_y, B_xB_s) \to P_\rho^{-l(y)} \subset (\overline{B_xB_s})^{-l(y)}$$

that is an isometry (up to a positive scalar) with respect to the local intersection form $(-,-)_y^{x,s}$ on the source and the Lefschetz form $(-,-)_\rho^{-l(y)}$ on the target.

As the restriction of a definite form is definite, and hence non-degenerate, by Lemma 2.3.8 we have

Lemma 2.3.15. *$S(< xs)$ and $HR(\underline{x}, s)$ imply $S_\pm(< xs, x, s)$.*

Combining Lemmas 2.3.8 and 2.3.15 and the uniqueness of invariant forms on indecomposable Soergel bimodules B_z in the presence of $S(z)$, we have

Lemma 2.3.16. *$S(< xs)$ and $HR(\underline{x}, s)$ imply $S(\leq xs)$ and $HR(\underline{xs})$.*

This is the core statement of the induction. What remains to show is that $S(\leq x)$ and $HR(\leq x)$ implies $HR(\underline{x}, s)$. In particular, this reduces Soergel's conjecture to a statement about the modules $\overline{B_x B_s}$ and their intersection forms, motivating the rest of this talk. In a later talk, we will need to consider a deformation of the Lefschetz operator ρ in order to complete the proof.

3. Invariant Forms on Soergel Bimodules

3.1. **Review and More Definitions.** Throughout this section, we'll fix a Coxeter system (W, S) and all the associated definitions from previous talks. In particular, m_{st} will be the order of st in W for simple reflections $s, t \in S$, the length function will be denoted l, we all know what expressions and reduced expressions are, Bruhat order is denoted $\leq$. As in the introduction, we'll fix a real reflection representation $\mathfrak{h}$ together with linearly independent subsets $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ ("roots") and $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$ ("coroots") such that

$$\alpha_s(\alpha_t^\vee) = -2\cos(\pi/m_{st}) \text{ for all } s, t \in S,$$

and we'll assume $\mathfrak{h}$ is minimal with respect to these properties. Remember that $s \in S$ acts on $\mathfrak{h}$ by

$$s.v = v - \alpha_s(v)\alpha_s^\vee$$

and that this extends to an action of W that is *reflection faithful*: it is faithful and induces a bijection between the "reflections" $\{wsw^{-1} : w \in W, s \in S\}$ in W and the codimension 1 fixed spaces in $\mathfrak{h}$. We also fix $\rho \in \mathfrak{h}^*$ as in the introduction, satisfying

$$\rho(\alpha_s^\vee) > 0 \text{ for all } s \in S.$$

We also have the Hecke algebra $\mathcal{H}$ of W over $\mathbb{Z}[v^{\pm 1}]$ as recalled in the introduction, with its standard basis $\{H_x\}_{x \in W}$, its Kazhdan-Lusztig basis $\{\underline{H}_x\}_{x \in W}$, its bar involution $\overline{p(v)H_x} := p(v^{-1})H_{x^{-1}}^{-1}$, and form $(H_x, H_y) = \delta_{x,y}$.

Remember that $B_s := R \otimes_{R^s} R(1)$ for $s \in S$ and for any expression $\underline{x} = (s_1, ..., s_n)$ we have the associated Bott-Samelson bimodule

$$BS(\underline{x}) := B_{s_1} \cdots B_{s_n} := B_{s_1} \otimes_R \cdots \otimes_R B_{s_n}.$$

Recall that B_s is graded free of rank 2 as a left OR right R-module with a basis for either action given by

$$c_{id} := 1 \otimes 1 \in B_s \qquad c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s).$$

By induction, it follows that $BS(\underline{x})$ is graded free of rank $2^{l(\underline{x})}$ with a basis given by the elements

$$c_{\underline{\epsilon}} := c_{\epsilon_{s_1}} \cdots c_{\epsilon_{s_n}}$$

for all possible choices of $\epsilon_{s_i} \in \{1, s_i\}$.

Definition 3.1.1. *Let $c_{bot} \in BS(\underline{x})$ denote the basis element*

$$c_{bot} := c_1 \cdots c_1$$

and let $c_{top} \in BS(\underline{x})$ denote the basis element

$$c_{top} := c_{s_1} \cdots c_{s_n}.$$

Let

$$Tr : BS(\underline{x}) \to R$$

be the trace defined by taking the coefficient of c_{top}.

Remember that we have the relations

$$rc_s = c_s r \tag{1}$$

$$rc_{id} = c_{id}(sr) + \partial_s(r)c_s \tag{2}$$

where $\partial_s : R \to R$ is the *Demazure operator* given by

$$\partial_s(r) := \frac{r - sr}{\alpha_s} \in R.$$

Recall that $BS(\underline{x}) = (R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R)(d)$. In particular, $BS(\underline{x})(-d)$ is a graded commutative ring with term-wise multiplication. We have the multiplication rules:

$$c_{id} \cdot c_{id} = c_{id} \tag{3}$$

$$c_{id} \cdot c_s = c_s \tag{4}$$

$$c_s \cdot c_s = c_s \alpha_s = \alpha_s c_s \tag{5}$$

Definition 3.1.2. *The* intersection form $\langle -, - \rangle_{BS(\underline{x})}$ *on $BS(\underline{x})$ is the invariant symmetric form defined by*

$$\langle b, b' \rangle_{BS(\underline{x})} = Tr(b \cdot b').$$

Denote the associated form on $\overline{BS(\underline{x})}$ by $\langle -, - \rangle_{\overline{BS(\underline{x})}}$. If $Tr_{\mathbb{R}}$ denotes the composition of Tr with the map $R \to R/R^+ = \mathbb{R}$, then this latter form is given by

$$\langle b, b' \rangle_{\overline{BS(\underline{x})}} = Tr_{\mathbb{R}}(b \cdot b').$$

Notice that any element $x \in \mathfrak{h}$ determines a Lefschetz operator (by left multiplication) on $\overline{BS(\underline{x})}$, because the form is invariant:

$$\langle x\alpha, \beta \rangle = \langle \alpha, x\beta \rangle.$$

Now let's define some operators on the BS bimodules (no offense to the bimodules [Lo]).

Definition 3.1.3. *Some definitions:*

(1) For $s \in S$, let

$$\mu : B_s \to R \qquad \mu(f \otimes g) = fg$$

be the multiplication operator.

(2) For an expression $\underline{x} = (s_1, ..., s_m)$ and index i, $1 \leq i \leq n$, let

$$\underline{x}_{\hat{i}} := (s_1, ..., \hat{s}_i, ..., s_m)$$

where the hat denotes omission

(3) Let

$$Br_i : BS(\underline{x}) \to BS(\underline{x})(2) \qquad b_1 \cdots b_m \mapsto b_1 \cdots (b_i c_{s_i}) \cdots b_m$$

(note that $b_i c_{s_i} = c_{s_i} b_i$ *because* $rc_s = c_s r$ *for all* $r \in R$*)*

(4) *Let*

$$\varphi_i : BS(\underline{x}) \to BS(\underline{x}_{\hat{i}}) \qquad b_1 \cdots b_m \mapsto b_1 \cdots \mu(b_i) \dots b_m$$

(5)

$$\chi_i : BS(\underline{x}_{\hat{i}}) \to BS(\underline{x}) \qquad b_1 \cdots b_{i-1} b_{i+1} \cdots b_m \mapsto b_1 \cdots b_{i-1} c_{s_i} b_{i+1} \dots b_m.$$

Remark 3.1.4. *By Equation (1) we have* $Br_i = \chi_i \circ \varphi_i$.

We'll need the following lemma:

Lemma 3.1.5. *As endomorphisms of* $BS(\underline{x})$, $\underline{x} = (s_1, ..., s_m)$, *we have*

$$\rho \cdot (-) = \sum_{i=1}^{m} (s_{i-1} \dots s_1 \rho)(\alpha_{s_i}^\vee) \chi_i \circ \varphi_i + (-) \cdot x^{-1} \rho.$$

Proof. This follows immediately from Equation (2) *draws nice diagram on the board*. ☐

3.2. **Construction and Properties of Invariant Forms.** Now we'll see how to inductively equip some R-bimodules with invariant forms. For an R-bimodule B, consider the two maps

$$\alpha, \beta : B \to BB_s = B \otimes_R B_s$$
$$\alpha(b) = bc_{id}$$
$$\beta(b) = bc_s.$$

WARNING: β is a morphism of bimodules, but α is only a morphism of left R-modules. By Equation (2) one has:

$$\alpha(br) = \alpha(b)(sr) + \beta(b)\partial_s(r).$$

Lemma 3.2.1. *Suppose that* B *is an* R*-bimodule, finitely generated graded free as a right* R*-module, and is equipped with an invariant form* $\langle -, - \rangle_B$. *Then* BB_s *is also finitely generated graded free as a right* R*-module (in fact* $BB_s \cong B \oplus B$ *as right or left* R*-modules - but not necessarily as bimodules!) and there is a unique invariant form* $\langle -, - \rangle_{BB_s}$ *on* BB_s*, which we call the* induced form*, satisfying*

$$\langle \alpha(b), \alpha(b') \rangle_{BB_s} = \partial_s(\langle b, b' \rangle_B)$$
$$\langle \alpha(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \text{ and } \langle \beta(b), \alpha(b') \rangle_{BB_s} = \langle b, b' \rangle_B$$
$$\langle \beta(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \alpha_s$$

for all $b, b' \in B$. *If* $\langle -, - \rangle_B$ *is nondegenerate, then so is* $\langle -, - \rangle_{BB_s}$. *If* $\langle -, - \rangle_B$ *is symmetric, then so is* $\langle -, - \rangle_{BB_s}$.

Furthermore, the intersection form on $BS(\underline{x})$ *is obtained in precisely this way, starting with the canonical multiplication form on* R *and iterating the induced form procedure.*

Proof. If $e_1, ..., e_m$ is a basis for B as a right R-module, then $\alpha(e_1), ..., \alpha(e_m), \beta(e_1), ..., \beta(e_m)$ is a basis for BB_s as a right R-module, so formulas above fix the form $\langle -, - \rangle_{BB_S}$ on this basis, giving uniqueness. Extending by right-R-linearity, it is then easy to see that the formulas above hold for all $b, b' \in B$; for example, the final formula is right-R-biliinear in b, b' so if it holds on a basis it holds everywhere; for the third one it's right-R-linear in b, so we just need to check if b, b' were in our basis and $r \in R$ we have

$$\langle \beta(b), \alpha(b'r) \rangle_{BB_s} = \langle \beta(b), \alpha(b')(sr) + \beta(b')\partial_s(r) \rangle_{BB_s}$$
$$:= \langle b, b' \rangle_B (sr) + \langle b, b' \rangle_B \alpha_s \partial_s(r) = \langle b, b' \rangle_B r = \langle b, b'r \rangle$$

where we used at some point that

$$sr + \alpha_s \partial_s(r) = r$$

which follows immediately from the definition of the Demazure operator. Similar calculations show that the other formulas hold and that the form is invariant. That the induced form is symmetric when the initial form is is clear.

For non-degeneracy, suppose the form is non-degenerate. Then let $e_1^*, ..., e_m^*$ be the dual basis for $e_1, ..., e_m$. Then

$$\alpha(e_1), ..., \alpha(e_m), \beta(e_1), ..., \beta(e_m)$$

and

$$\beta(e_1^*), ..., \beta(e_m^*), \alpha(e_1^*), ..., \alpha(e_m^*)$$

are right-R-bases for BB_s, and in these bases the form is represented by the matrix

$$\begin{pmatrix} I & \alpha_s I \\ 0 & I \end{pmatrix}$$

(the zero block in the lower left arises because $\partial_s(1) = 0$), which is visibly invertible, so $\langle -, - \rangle_{BB_s}$ is non-degenerate.

That the intersection form on $BS(\underline{x})$ arises by an iteration of this procedure is proved by induction and is straightforward (look at the $c_{\underline{\epsilon}}$-basis). □

Corollary 3.2.2. *The intersection form on a Bott-Samelson bimodule is non-degenerate.*

Lemma 3.2.3. *The Lefschetz form* $(-,-)_\rho^{-l(x)}$ *on* $\overline{BS(\underline{x})}^{-l(x)} \cong \mathbb{R}$ *is positive definite, when* $\underline{x}$ *is a reduced expression.*

Proof. Recall that $c_{bot} := c_{id} \cdots c_{id}$ spans $BS(\underline{x})^{-l(x)}$. We claim that

$$\rho^{l(x)} c_{bot} = N c_{top} \in \overline{BS(\underline{x})}$$

for some $N > 0$, which will imply the result. The proof of this is by induction on $l(\underline{x})$. The case $l(\underline{x}) = 0$ is clear.

By Lemma 3.1.5 we have

$$\rho \cdot c_{bot} = \sum_i (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) \chi_i(c_{bot}) + c_{bot} \cdot (x^{-1}\rho)$$

(notice that the c_{bot} in the sum lives in $BS(\underline{x}_{\hat{i}})$). Notice that $(s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee)$ is positive for all i by our assumption on ρ and the fact that $\underline{x}$ is a reduced expression. The last term $c_{bot} \cdot (x^{-1}\rho)$ is obviously zero in $\overline{BS(\underline{x})}$, so it suffices to know that $\rho^{l(x)-1}\chi_i(c_{bot}) = N_i c_{top}$ in $\overline{BS(\underline{x})}$ for some $N_i \geq 0$ and that at least one N_i is strictly positive.

There are two cases:

Case 1: $\underline{x}_{\hat{i}}$ is a reduced expression. Then by induction

$$\rho^{l(x)-1} c_{bot} = N_i c_{top} \in \overline{BS(\underline{x}_{\hat{i}})}$$

for some $N_i > 0$. Clearly $\chi_i(c_{top}) = c_{top}$, so $\rho^{l(x)-1}\chi_i(c_{bot}) = N_i c_{top} \in \overline{BS(\underline{x})}$.

Case 2: $\underline{x}_{\hat{i}}$ is not a reduced expression. In this case we have

$$BS(\underline{x}_{\hat{i}}) \cong \bigoplus B_z^{\oplus p_z}$$

with all z appearing on the righthand side satisfying $l(z) < l(x) - 1$ and $p_z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. For degree reasons $\rho^{l(x)-1}$ vanishes on any such $\overline{B_z}$ (remember that ρ is a degree 2 operator and that $\overline{B_z}$ is supported in degrees d with $|d| \leq l(z) < l(x) - 1$), and therefore vanishes identically on $\overline{BS(\underline{x}_{\hat{i}})}$. In particular, $\rho^{l(x)-1}\chi_i(c_{bot}) = 0$ for such i.

Therefore we have

$$\rho^{l(x)} c_{bot} = \left(\sum_i (s_{i-1} \cdots s_i \rho)(\alpha_i^\vee) N_i \right) c_{top} \in \overline{BS(\underline{x})}$$

with

$$\sum_i (s_{i-1} \cdots s_i \rho)(\alpha_i^\vee) N_i > 0$$

as needed. □

Finally, we make the following observation as promised in the introduction:

Lemma 3.2.4. *If $S(x)$ holds, then $HR(x)$ and $HR(\underline{x})$ are equivalent, for any reduced expression $\underline{x}$ for x.*

Proof. When $S(x)$ holds, then by definition the intersection form on B_x is a non-degenerate invariant form that gives a positive Lefschetz form in the lowest graded space. But from the previous lemma, we see that the restriction of the intersection form on $BS(\underline{x})$ to $B_x \subset BS(\underline{x})$ (embedding as summand) is also such a form. As the space of invariant forms is 1-dimensional, we see that these forms agree up to a positive scalar multiple. So then the Hodge-Riemann bilinear relations are clearly equivalent for the two forms. □

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