HIGHEST WEIGHT CATEGORIES

Let $\mathbb{F}$ be a field, Λ be a finite poset and $\mathcal{C}$ be an $\mathbb{F}$-linear artinian category with simple objects parameterized by Λ. By a highest weight category structure on $\mathcal{C}$ with respect to Λ we mean a collection of *standard* objects $\Delta(\lambda)$ in $\mathcal{C}$, one for each $\lambda \in \Lambda$ such that

- $\mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leqslant \mu$.
- $\mathrm{End}(\Delta(\lambda)) = \mathbb{F}$ for any λ.
- $\mathcal{C}$ has enough projectives. The indecomposable projectives are parameterized by $\Lambda, \lambda \mapsto P(\lambda)$, where $P(\lambda)$ is the projective cover of $L(\lambda)$. The object $P(\lambda)$ surjects onto $\Delta(\lambda)$ and the kernel admits a filtration whose quotients are of the form $\Delta(\mu)$ with $\mu > \lambda$.

We write $\mathcal{C}^\Delta$ for the full subcategory of all standardly filtered objects in $\mathcal{C}$, i.e., all objects that admit a filtration with standard quotients.

0.1. Simple constituents of standards.
Let $L(\lambda)$ denote the simple corresponding to $\lambda \in \Lambda$. Show that if $L(\lambda)$ occurs in $\Delta(\mu)$, then $\lambda \leqslant \mu$. Moreover, show that the multiplicity of $L(\lambda)$ in $\Delta(\lambda)$ is 1, and $\Delta(\lambda) \twoheadrightarrow L(\lambda)$.

0.2. Ext's between standards.
Show that if $\mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$, then $\lambda < \mu$.

0.3. Subcategories.
Let $\Lambda' \subset \Lambda$ be an ideal in the sense that if $\lambda \in \Lambda'$ and $\mu \leqslant \lambda$, then $\mu \in \Lambda'$. Consider the Serre subcategory $\mathcal{C}' \subset \mathcal{C}$ spanned by the simples $L(\lambda)$ with $\lambda \in \Lambda'$.

(1) Show that $\mathcal{C}'$ is a highest weight category with standard objects $\Delta(\lambda), \lambda \in \Lambda'$.
(2) Show that the left adjoint functor $\iota^!$ to the embedding $\mathcal{C}' \hookrightarrow \mathcal{C}$ is exact on $\mathcal{C}^\Delta$.
(3) Deduce that there is a projective resolution $P^\bullet$ in $\mathcal{C}$ of $M \in \mathcal{C}'^\Delta$ such that $\iota^!(P^\bullet)$ is a projective resolution of M in $\mathcal{C}'$ (in fact, this is true for any projective resolution).

0.4. Quotients.
Now let $\mathcal{C}'' := \mathcal{C}/\mathcal{C}'$ and π be the quotient functor $\mathcal{C} \twoheadrightarrow \mathcal{C}'$. Let $\pi^!$ denote its left adjoint functor.

(1) Show that $\mathcal{C}''$ is a highest weight category with standard objects $\pi(\Delta(\lambda))$.
(2) Moreover, show that the natural morphism $\pi^!(\pi(M)) \to M$ is an isomorphism provided M admits a filtration whose quotients are $\Delta(\mu)$ with $\mu \notin \Lambda'$.
(3) Deduce that $\pi^!$ gives rise to an equivalence between $\mathcal{C}''^\Delta$ and a full subcategory in $\mathcal{C}^\Delta$ consisting of all objects whose filtration quotients are $\Delta(\mu)$ with $\mu \notin \Lambda'$.

0.5. Characterization of projectives.
Show that a standardly filtered object $P \in \mathcal{C}$ is projective if and only if $\mathrm{Ext}^1(P, \Delta(\lambda)) = 0$ for all $\lambda \in \Lambda$.

0.6. Axiomatic characterization of standards.
Show that the objects $\Delta(\lambda)$ are uniquely recovered from the poset structure on Λ as follows. Pick $\lambda \in \Lambda$. Let $\Lambda_{\leqslant \lambda} = \{\mu \in \Lambda | \mu \leqslant \lambda\}$. Consider the Serre subcategory $\mathcal{C}_{\leqslant \lambda}$ spanned by the simples $L(\mu)$ with $\mu \in \Lambda_{\leqslant \lambda}$. Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathcal{C}_{\leqslant \lambda}$.

0.7. Costandard objects.
Let $\nabla(\lambda)$ stand for the injective hull of $L(\lambda)$ in $\mathcal{C}_{\leqslant \lambda}$. This is a so called *costandard object*. Show that $\dim \mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$, while $\mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$ for $i > 0$.

0.8. **Highest weight structure on $\mathcal{C}^{opp}$.** Prove that the collection of costandard objects $\nabla(\lambda)$ makes the opposite category $\mathcal{C}^{opp}$ into a highest weight category with respect to the poset Λ.

0.9. **Characterization of (co)standardly filtered objects.** Prove the object $M \in \mathcal{C}$ is Δ-filtered if and only if $\mathrm{Ext}^i(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda, i > 0$, if and only if $\mathrm{Ext}^1(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$. Deduce the kernel of an epimorphism of standardly filtered objects is standardly filtered. State and prove the dual statements.

0.10. **BGG reciprocity.** Show that the multiplicity of $\Delta(\lambda)$ in $P(\mu)$ coincides with the multiplicity of $L(\mu)$ in $\nabla(\lambda)$.

0.11. **Tilting objects.** An object in $\mathcal{C}$ is called *tilting* if it is both standardly and costandardly filtered.

  (1) For $\lambda \in \Lambda$ consider an object $T(\lambda)$ constructed as follows. Order linearly elements of $\{\mu \in \Lambda | \mu \leqslant \lambda\}$ refining the original poset structure, say $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_k$. Construct the object $T^i(\lambda), i = 1, \ldots, k$ inductively as follows. Set $T^1(\lambda) = \Delta(\lambda)$. Further, if $T^{i-1}(\lambda)$ is already defined let $T^i(\lambda)$ be the extension of $\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i)$ by $T^{i-1}(\lambda)$ corresponding to the unit endomorphism of $\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda))$. Show that $T(\lambda) := T^k(\lambda)$ is an indecomposable tilting.
  (2) Prove that any other tilting in $\mathcal{C}$ is isomorphic to the direct sum of the objects $T(\lambda)$.

0.12. **Ringel duality.** Set $T := \bigoplus_{\lambda \in \Lambda} T(\lambda)$. Let $\mathcal{C}^\vee$ be the category of finitely generated $\mathrm{End}(T)$-modules. Show that this category is highest weight with respect to the opposite poset Λ^{opp} with standard objects $\mathrm{Hom}(\Delta(\lambda), T)$. Show that $(\mathcal{C}^\vee)^\Delta \cong (\mathcal{C}^\Delta)^{opp}$ and that, under this identification, the projective objects in $\mathcal{C}$ correspond to tilting objects in $\mathcal{C}^\vee$, while tilting objects in $\mathcal{C}$ correspond to projective objects in $\mathcal{C}^\vee$. Finally, identify $(\mathcal{C}^\vee)^\vee$ with $\mathcal{C}$.