9. Commutativity and centers

Exercise 9.1. Show that \( \{ \cdot, \cdot \}_{t,c} = t \{ \cdot, \cdot \} \), where \( \{ \cdot, \cdot \} \) is the standard bracket on \( S(V)^\Gamma \).

Exercise 9.2. Prove the commutativity theorem in the case when \( V \) is not necessarily symplectically irreducible.

Exercise 9.3. Let \( \mathcal{A} \) be a \( \mathbb{Z}_{>0} \)-filtered algebra. If \( \text{gr} \mathcal{A} \) is finitely generated, then so is \( \mathcal{A} \).

Problem 9.1. Let \( p \in \mathcal{P}_0 \). Equip \( Z_p \) with a filtration restricted from \( H_p \). Show that \( \text{gr} Z_p = S(V)^\Gamma \). Deduce that \( H_p \) is a finitely generated module over \( Z_p \).

Problem 9.2. Now let \( p \notin \mathcal{P}_0 \). Show that the center of \( H_p \) coincides with \( \mathbb{C} \) as follows:

1. Let \( z \) lie in the center of \( H_p \). Show that \( \text{gr} z \in \text{gr} H_p = S(V)^\# \Gamma \) actually lies in \( S(V)^\Gamma \).

2. Show that \( \text{gr} z \) lies in the Poisson center of \( S(V)^\Gamma \), meaning that \( \{ \text{gr} z, S(V)^\Gamma \} = 0 \).

3. Show that the Poisson center of \( S(V)^\Gamma \) coincides with \( \mathbb{C} \).

Problem 9.3. In this problem we are going to equip \( Z_c \) with a structure of a Poisson algebra. Fix \( c \) and consider \( H_{t,c} \) as an algebra over \( \mathbb{C}[t] \) by making \( t \) an independent variable.

1. Let \( a, b \in Z_c \). Lift \( a, b \in H_c = H_{t,c}/(t) \) to elements \( \tilde{a}, \tilde{b} \in H_{t,c} \). Show that \( [\tilde{a}, \tilde{b}] \in tH_{t,c} \) and that the element \( \frac{1}{t}[\tilde{a}, \tilde{b}] \) modulo \( t \) depends only on \( a, b \) and lies in \( Z_c \). Let \( \{ a, b \} \) be that element. Show that \( \{ \cdot, \cdot \} \) is the Poisson bracket.

2. Show that \( \{ Z_c^i, Z_c^j \} \subset Z_c^{i+j-2} \). Show that the induced bracket on \( \text{gr} Z_c = S(V)^\Gamma \) is a nonzero multiple of the standard bracket. Can you identify the scalar factor?

Problem 9.4. Show that the scheme \( C_p \) is irreducible and normal (and, well, Cohen-Macaulay and Gorenstein, if you know what these words mean).

Problem 9.5. Show that if \( C_p \) is smooth, then \( H_p e \) is a locally free \( H_p \)-module.

Problem 9.6. Let \( \mathcal{A} \) be a filtered algebra. Show that if \( \text{gr} \mathcal{A} \) has finite global dimension, then \( \mathcal{A} \) does.

Problem 9.7. Prove that if \( C_p \) is smooth, then \( p \) is spherical (we deal here with \( p \in \mathcal{P}_0 \)).