3. McKay correspondence upgraded (from last time)

Exercise 3.3. A map $\mathbb{C}^2 \otimes \mathbb{C} \Gamma \rightarrow \mathbb{C} \Gamma$ extends to a representation from $\text{Rep}_\Gamma(\mathbb{C}(x, y)\# \Gamma; \mathbb{C} \Gamma)$ if and only if it is $\Gamma$-equivariant.

Exercise 3.4. Show that

$$\text{Hom}_\Gamma(\mathbb{C}^2 \otimes \mathbb{C} \Gamma, \mathbb{C} \Gamma) = \bigoplus_{i,j=0}^r M_{ij} \otimes \text{Hom}_\mathbb{C}(N_i^*, N_j^*)$$

$$= \bigoplus_{i,j=0}^r \text{Hom}_\mathbb{C}(N_i^*, N_j^*)^{\otimes m_{ij}} = \bigoplus_{i,j=0}^r \text{Hom}_\mathbb{C}(\mathbb{C}^{h_i}, \mathbb{C}^{h_j})^{m_{ij}}.$$

Note that the first equality is canonical, the second depends on the choice of a basis in $M_{ij}$, while the third depends on the choice of bases in $N_i^*$.

4. Deformed preprojective algebras

Exercise 4.1. Show that $CQ$ is associative and $\sum_{i \in Q_0} \epsilon_i$ is a unit in $CQ$. Further, show that, as a unital associative algebra, $CQ$ is generated by $\epsilon_i, i \in Q_0$, and $a \in Q_1$ subject to the relations $\epsilon_i\epsilon_j = \delta_{ij}\epsilon_i, \sum_{i \in Q_0} \epsilon_i = 1, \epsilon_ia = \delta_{ih(a)}a, a\epsilon_i = \delta_{id(a)}a$.

Exercise 4.2. Use the universal properties of all algebras involved to show that $\mathbb{C}(x, y)\# \Gamma \cong T_{C\Gamma}(\mathbb{C}^2 \otimes \mathbb{C} \Gamma)$ and $CQ \cong T_{(CQ)^0}(CQ)^1$.

Exercise 4.3. Let $A$ be an associative algebra, and $e \in A$ be an idempotent. We define functors $\pi : A\text{-Mod} \rightarrow eAe\text{-Mod}$ by $\pi(M) = eM$, and $\pi^1 : eAe\text{-Mod} \rightarrow A\text{-Mod}$ by $\pi^1(N) = Ae \otimes_{eAe} N$.

- Show that $\pi$ is an exact functor, that $\pi$ can be written as $M \mapsto eA \otimes_A M$, and that $\pi^1$ is left adjoint to $\pi$.
- Suppose that $eAe = A$. Check that if $\pi(M) = 0$, then $M = 0$. Further check that the natural homomorphism $eA \otimes_{eAe} eM \rightarrow M$ is surjective. Finally, show that $Ae \otimes_{eAe} eM \rightarrow M$ is injective by applying $\pi$.
- Deduce that $Ae \otimes_{eAe} eA = A$ as a bimodule.

Exercise 4.4. Suppose $e$ is an idempotent in $A$ such that $eAe = A$. Show that the functor $M \mapsto eMe$ is an equivalence between the categories of $A$ and $eAe$-bimodules intertwining the tensor products (meaning that $e(M \otimes A N)e = eMe \otimes_{eAe} eNe$). Deduce that $eT_A(M)e$ is naturally identified with $T_{eAe}(eMe)$.

Exercise 4.5. Check that the maps $\text{Hom}(M, \mathbb{C}^2 \otimes M') \rightarrow \text{Hom}(\mathbb{C}^2 \otimes M, M'), \psi \mapsto (\omega \otimes 1_M) \circ (1_{\mathbb{C}^2} \otimes \psi)$ and $\text{Hom}(\mathbb{C}^2 \otimes M, M') \rightarrow \text{Hom}(M, \mathbb{C}^2 \otimes M'), \varphi \mapsto (1_{\mathbb{C}^2} \otimes \varphi) \circ (\zeta \otimes 1_M)$ are inverse to each other.

Problem 4.1. Prove the CBH lemma in the cyclic case, assuming that the orientation on $Q$ is also cyclic. Hint: for $x, y$ we can take $\Gamma$-eigenvectors.