PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

2. CBH algebras (left from last time)

Exercise 2.5. Prove that there are no non-constant invariant polynomials for the action of the one-dimensional torus $\mathbb{C}^\times$ on $\mathbb{C}^n$ given by $t.(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$.

Exercise 2.6. Use the theorem (the only statement called this way in the lecture) to show that the closure of any orbit of a reductive group action on an affine variety contains a unique closed orbit.

Problem 2.6. Show that the algebra of invariants $\mathbb{C}[X]^G$, where $X = \text{Mat}_n(\mathbb{C})$ and $G = \text{GL}_n(\mathbb{C})$ acts on $X$ by conjugations, is generated by the coefficients of the characteristic polynomial of a matrix and is isomorphic to the algebra of polynomials in $n$ variables. A hint: consider the restriction to the subspace of diagonal matrices.

Problem 2.7. In the setting of the previous problem, check directly that every fiber indeed contains a single closed orbit and that this orbit consists of diagonalizable matrices.

3. McKay correspondence upgraded

Exercise 3.1. Show that if the $\text{GL}_N(\mathbb{C})$-orbit of a representation in $\text{Rep}(\mathcal{A}, N)$ is closed, then the representation is semisimple.

Problem 3.1. Use the Hilbert-Mumford criterium to show that the orbit of a semisimple representation is closed.

Exercise 3.2. Show that the stabilizer in $\Gamma$ of any nonzero point in $\mathbb{C}^2$ is trivial.

Exercise 3.3. A map $\mathbb{C}^2 \otimes \Gamma \rightarrow \Gamma$ extends to a representation from $\text{Rep}_\Gamma(\mathbb{C}(x, y)^\#_\Gamma, \mathbb{C})$ if and only if it is $\Gamma$-equivariant.

Exercise 3.4. Set $M_{ij} := \text{Hom}_\Gamma(\mathbb{C}^2 \otimes \Gamma, \mathbb{C})$. Show that

$$\text{Hom}_\Gamma(\mathbb{C}^2 \otimes \Gamma, \mathbb{C}) = \bigoplus_{i,j=0}^r M_{ij} \otimes \text{Hom}_\mathbb{C}(N_i^*, N_j^*) = \bigoplus_{i,j=0}^r \text{Hom}_\mathbb{C}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j})^{m_{ij}}.$$