PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

17. Procesi bundles and their deformations

**Problem 17.1.** Let $X$ be an algebraic variety, $\mathcal{F}_0$ be a coherent sheaf on $X$ and $\mathcal{D}$ be a FCS (=flat, complete and separated) deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar]]$.

(1) Show that the category of finitely generated modules (i.e., sheaves) over $\mathcal{D}/(\hbar^n)$ has enough injective objects. How are the injectives for different $n$ related?

(2) Show that if $\operatorname{Ext}^2(\mathcal{F}_0, \mathcal{F}_0) = 0$, then there exists a flat deformation of $\mathcal{F}_0$ to a right module $\mathcal{F}_n$ over $\mathcal{D}/(\hbar^{n+1})$. Moreover, show that these deformations may be chosen in a compatible way and so give rise to a FCS deformation $\mathcal{F}$ of $\mathcal{F}_0$ to a right module over $\mathcal{D}$.

(3) Finally, show that if $\operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F}_0) = 0$, then all the deformations above are unique.

**Exercise 17.1.** Let $V_1, V_2$ are $\mathbb{C}[[\mathfrak{g}, \hbar]]$-modules that are flat, complete and separated. Let $\iota : V_1 \to V_2$ be a $\mathbb{C}[[\mathfrak{g}, \hbar]]$-module homomorphism that is an isomorphism modulo $(\mathfrak{g}, \hbar)$. Show that $\iota$ is an isomorphism.

**Exercise 17.2.** Show that any fiber of a Procesi bundle is isomorphic to $\mathbb{C} \Gamma_n$ as a $\Gamma_n$-module.

**Problem 17.2.** Show that the dual of a Procesi bundle is again a Procesi bundle.

**Exercise 17.3.** Let $A_0$ be a $\mathbb{Z}_{>0}$-graded vector space and $A$ be its FCS deformation over $\mathbb{C}[[x_1, \ldots, x_n]]$. Equip $A$ with a $\mathbb{C}^\times$-action such that $t.(x_i.a) = t^2 x_i t.a$ and the projection $A \to A_0$ is $\mathbb{C}^\times$-equivariant (where the action of $\mathbb{C}^\times$ on the $i$th component of $A_0$ is by $t \mapsto t^i$). Show that the $\mathbb{C}^\times$-finite part of $A$ is a graded deformation of $A_0$ over $\mathbb{C}[x_1, \ldots, x_n]$.

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1 Also appeared last time