15. Quotient singularities as quiver varieties

The purpose of this problem set is to recover results of Lecture 15 in the special case of \( \Gamma_1 = \{1\} \).

**Problem 15.1.** Describe the \( GL(n) \)-orbits on \( \text{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n \). Deduce that there are \( n + 1 \) components in \( \mu^{-1}(0) \) and that they all have codimension \( n^2 \).

As in the correction to the lecture, we show that \( \mu^{-1}(0) \) is reduced. So \( \mu^{-1}(0)/G \) is a variety.

**Problem 15.2.** This problem establishes a bijective morphism \( \psi : \mathbb{C}^{2n}/S_n \to \mu^{-1}(0)/G \).

1. Let \( A, B \in \text{Mat}_n(\mathbb{C}) \) be such that \( \text{rk}[A, B] \leq 1 \). Show that, in some basis, \( A, B \) are upper triangular.

2. Deduce that any irreducible representation of \( \Pi^0(Q^{CM}) \) is actually a one-dimensional representation of \( \Pi^0(Q^{MK}) \).

3. Use this to produce a required morphism.

It remains to show that \( \psi^* \) is isomorphism of algebras.

**Problem 15.3.** Show that \( \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n} \) is generated by the polynomials of the form \( \sum_{i=1}^{n} (x_i + ty_i)^n \). Deduce that \( \psi^* \) is surjective and, using this, that \( \psi^* \) is an isomorphism.