14. Quantum Hamiltonian reduction vs SRA

Exercise 14.1. Prove that $\Phi(\xi, \eta) = \frac{1}{h} [\Phi(\xi), \Phi(\eta)]$ for any $\xi, \eta \in \mathfrak{g}$.

Exercise 14.2. Prove that the center of $W_h(V)$ coincides with $\mathbb{C}[\hbar]$.

Exercise 14.3. Describe the map $\xi \mapsto \xi_A$ for $A_h = D_h(X_0)$ and show that $\xi \mapsto \xi_{X_0}$ is a quantum comoment map.

Problem 14.1. Let $X_0$ be a vector space equipped with a linear action of a group $G$. Then $W_h(X_0 \oplus X_0^*)$ is the same algebra as $D_h(X_0)$. We get two quantum comoments maps, $\Phi_D, \Phi_W$ for the $G$-action on this algebra. Describe the difference $\Phi_D - \Phi_W$.

Exercise 14.4. Let $A_h$ be an associative unital algebra over $\mathbb{C}[\hbar]$, flat over $\mathbb{C}[\hbar]$, complete and separated in the $\hbar$-adic topology, and such that $A := A_h/(\hbar) = \mathbb{C}$ is commutative. Let $S$ be a multiplicatively closed subset of $A$ that does not contain 0 and let $\pi_X$ denote the projection $A_h/(\hbar^k) \to A$. Show that $\pi^{-1}(S)$ satisfies the Ore condition: i.e., for all $a \in A_h/(\hbar^k), s \in \pi^{-1}(S)$, there are $a' \in A_h/(\hbar^k), s' \in \pi^{-1}(S)$ such that $as' = a's$. Show that there are natural epimorphisms $A_h/(\hbar^{k+1})[\pi_k(S)^{-1}] \to A_h/(\hbar^k)[\pi_k(S)^{-1}]$ and prove that $A_h[S^{-1}] := \lim_{\leftarrow k} A_h/(\hbar^k)[\pi_k(S)^{-1}]$ is flat over $\mathbb{C}[[\hbar]]$.

Exercise 14.5. Show that the product on $A_h/\mathbb{Z}G$ is well-defined.

Exercise 14.6. Check that the image of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ in $[A_h/A_h\Phi([\mathfrak{g}, \mathfrak{g}])$ consists of $G$-invariant elements that commute with $[A_h/A_h\Phi([\mathfrak{g}, \mathfrak{g}]])^G$.

Problem 14.2. Let $G$ be a reductive group acting freely on a smooth affine variety $X_0$. Identify $D_h(X_0)/\mathbb{Z}G$ with $D_h(X_0)/G$. 
