13. Quantum CM systems and Rational Cherednik algebras

Exercise 13.1. Prove that \( \omega D_{\omega} \omega^{-1} = D_{\omega} \) for all \( \omega \in W, a \in \frak{h} \).

Exercise 13.2. Prove an analog of Proposition on the properties of Dunkl operators for complex reflection groups.

Exercise 13.3. Let \( W \) be a complex reflection group.

1. Show that \( \text{ad} f \) is a locally nilpotent operator on \( H_{\text{Reg}} \) for any \( f \in \mathbb{C}[\frak{h}]^W \).
2. Deduce that the localization \( H_{\text{Reg}}[\delta^{-1}] \) exists.
3. Show that the Dunkl homomorphism \( H_{\text{Reg}} \to D_{\text{Reg}} \) factors through a unique homomorphism \( H_{\text{Reg}}[\delta^{-1}] \to D_{\text{Reg}} \).
4. Show that the homomorphism \( H_{\text{Reg}}[\delta^{-1}] \to D_{\text{Reg}} \) is an isomorphism.

Exercise 13.4. Prove that the Dunkl homomorphism \( \Theta \) is injective modulo \( \mathbb{C} \) and hence is injective.

Exercise 13.5. Define \( \varphi \) to be the identity on \( \mathbb{C}[^{\text{Reg}}_{\text{Reg}}] \) and \( \varphi(a) = a + \sum_{s \in S} c_s \langle a, s \rangle s \).
Show that \( \varphi \) extends to a \( \mathbb{C}[\frak{h}] \)-linear automorphism of \( D_{\text{Reg}} \).

Problem 13.1. Let \( \Gamma = \frak{S}_n \) and \( \frak{h} = \mathbb{C}^n \) (and not the reflection representation, this is a minor technicality). The goal of this problem will be to relate the CM space \( C \) to the Spec\( (eH_{0,e}) \). We are going to produce a morphism \( \text{Rep}_T(H_{0,e}, \text{CG})/\text{GL}((\text{CG})^T \to C \), to show that this is an isomorphism. Then we prove that the natural morphism \( \text{Rep}_T(H_{0,e}, \text{CG})/\text{GL}((\text{CG})^T \to \text{Spec}(eH_{0,e})) \) is an isomorphism.

Let \( y_1, \ldots, y_n \) be the tautological basis in \( \mathbb{C}^n = \frak{h} \) and \( x_1, \ldots, x_n \) be the dual basis in \( \frak{h}^* \). The elements \( x_n, y_n \) still act on \( N^\frak{s}_{n-1} \cong \mathbb{C}^n \). Show that \( [x_n, y_n] \in O = \{ A \mid \text{tr} A = 0, \rk(A + E) = 1 \} \) for a suitable choice of \( c \). Deduce that we have a morphism \( \text{Rep}_T(H_{0,e}, \text{CG}) \to \mu^{-1}(O) \).
Show that it descends to a morphism \( \text{Rep}_T(H_{0,e}, \text{CG})/\text{GL}((\text{CG})^T \to C \). Show that the latter is finite and birational. Deduce that it is an isomorphism.

Show that a natural morphism \( \text{Rep}_T(H_{0,e}, \text{CG})/\text{GL}((\text{CG})^T \to \text{Spec}(eH_{0,e}) \) (how is it constructed, by the way?) is also finite and birational. Deduce that it is an isomorphism.