1. Kleinian singularities

**Problem 1.1.** Let $G$ be a finite subgroup of $SO_3(\mathbb{R})$. Consider its action on the unit sphere. Show that any non-unit element of $G$ fixes a unique pair of opposite points and that the stabilizer of each point $P$ is cyclic of some order, say, $n_P$. Choose representatives $P_1, \ldots, P_k$ of orbits with non-trivial stabilizers, one in each orbit. Show that

$$2\left(1 - \frac{1}{n_1}ight) = \sum_{i=1}^{k} \left(1 - \frac{1}{n_{P_i}}\right).$$

Use this to show that the finite subgroups of $SO_3(\mathbb{R})$ are precisely the following:

1) The cyclic group of order $n$ – generated by a rotation by the angle of $2\pi/n$.
2) The dihedral group of order $2n$ with $n \geq 2$: the group of rotational symmetries of a regular $n$-gon on the plane inside of the 3D space (a regular 2-gon = a segment).
3) The group of rotational symmetries of the regular tetrahedron isomorphic to the alternating group $A_4$.
4) The group of rotational symmetries of the regular cube/octahedron isomorphic to the symmetric group $S_4$.
5) The group of rotational symmetries of the regular dodecahedron/icosahedron isomorphic to $A_5$.

**Problem 1.2.** Use the previous problem to deduce that the complete list of finite subgroups $SL_2(\mathbb{C})$ up to conjugacy is as follows.

$(A_r)$ The cyclic group of order $r + 1$, i.e., $\left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} | \epsilon^{r+1} = 1 \right\}$.

$(D_r)$ The dihedral group of order $4(r - 2), r \geq 4$: $\left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} | \epsilon^{2(r-2)} = 1 \right\}$.

$(E_6)$ The double cover of $A_4 \subset SO_3(\mathbb{R})$.

$(E_7)$ The double cover of $S_4 \subset SO_3(\mathbb{R})$.

$(E_8)$ The double cover of $A_5 \subset SO_3(\mathbb{R})$.

**Problem 1.3.** Compute the McKay graph for $\Gamma \subset SL_2(\mathbb{C})$ of type $D_r$.

**Problem 1.4.** This problem discusses the Kleinian group of type $E_6$.

1) We start with a construction. Take the group $Q_8$ of unit quaternions. It has elements $\{\pm 1, \pm i, \pm j, \pm k\}$. Show that the cyclic group $\mathbb{Z}_3$ acts on $Q_8$ by automorphisms in such a way that the generator $\omega$ acts as follows: $\omega(-1) = -1, \omega(i) = j, \omega(j) = k, \omega(k) = i$. Embed the semi-direct product $\Gamma := \mathbb{Z}_3 \ltimes Q_8$ into $SL_2(\mathbb{C})$. Further, show that $\Gamma/\{\pm 1\} \cong A_4$.

2) Show that $\Gamma$ has 3 one-dimensional, 3 two-dimensional and 1 three-dimensional irreducible representations.

3) Compute the McKay graph for $\Gamma$.

**Problem 1.5.** Show that for $\Gamma \subset SL_2(\mathbb{C})$ of type $D_r$, we have $\mathbb{C}[x, y]^{\Gamma} / \cong \mathbb{C}[x_1, x_2, x_3] / (x_1^{-r+1} + x_1 x_2^2 + x_3^2)$. 