8. Spherical SRA

Let $V$ be a finite dimensional vector space equipped with a symplectic form $\omega$ and $\Gamma$ be a finite subgroup in $\text{Sp}(V)$. Let $S$ denote the subset of all symplectic reflections $s \in \Gamma$, i.e., all elements with $\text{rk}(s - 1_V) = 2$ and let $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_m$ be the decomposition of $S$ into conjugacy classes. To each $s \in S$ we assign the form $\omega_s \in \bigwedge^2 V^*$ by

$$\omega_s(u, v) = \begin{cases} \omega(u, v), & u, v \in \text{im}(s - 1_V), \\ 0, & u \in \text{ker}(s - 1_V). \end{cases}$$

In Lecture 6 we have introduced the universal SRA. We pick independent variables $t, c_1, \ldots, c_m$ and consider the vector space $P$ with basis $t, c_1, \ldots, c_m$. Then we define the algebra $H$ by

$$H = S(P) \otimes T(V)^\#\Gamma/(u \otimes v - v \otimes u - t\omega(u, v) - \sum_{i=1}^m c_i \sum_{s \in S_i} \omega_s(u, v)s \mid u, v \in V).$$

Our goal is to prove that $H$ is a graded deformation of $S(V)^\#\Gamma$, meaning $H$ is free as an $S(P)$-module.

In Lecture 7 we have computed some Hochschild cohomology of $S(V)^\#\Gamma$. Namely, we have seen that $\text{HH}^2(S(V)^\#\Gamma)^i = 0$ for $i \leq -3$, and $\text{HH}^3(S(V)^\#\Gamma)^i = 0$ for $i \leq -4$. This implies that there is a universal graded deformation $H_{un}$ of $S(V)^\#\Gamma$ over $S(P_{un})$, where $P_{un} := (\text{HH}^2(S(V)^\#\Gamma)^{-2})^*$. We have computed $P_{un}$ in the case when $V$ is symplectically irreducible as $\Gamma$-module, we have found that $\text{dim} P_{un} = m + 1$. In fact, in general, $\text{dim} P_{un} = m + \text{dim}(\bigwedge^2 V)^\Gamma$, this is proved completely analogously to the last exercise in the previous lecture. The universality means that for any other graded deformation $\tilde{H}$ of $S(V)^\#\Gamma$ over $S(\tilde{P})$ there is a unique linear map $P_{un} \to \tilde{P}$ such that there is an $S(\tilde{P})$-linear isomorphism $S(\tilde{P}) \otimes_{S(P_{un})} H_{un} \cong \tilde{H}$ that is the identity modulo $\tilde{P}$.

The first thing we will do in this lecture: we will identify $H_{un}$ and $H$ in the case when $V$ is symplectically irreducible. This will easily imply that $H$ is a deformation, in general. We will also see that the isomorphism in the end of the previous paragraph is unique that makes $H_{un}$ a universal object in the categorical sense.

Our original goal was to study the deformations of the invariant subalgebra $S(V)^\Gamma$. We get a deformation $\epsilon H \epsilon$ over $S(P)$ (it is almost for sure is not universal in the categorical sense; in general it is unknown whether this exhausts all reasonable deformations). We will study an interplay between $H$ and $\epsilon H \epsilon$. This interplay is provided by the bimodules $H \epsilon$ and $\epsilon H$. We will see that $\epsilon H \epsilon$ and $H$ are mutual centralizers of each other in these bimodules (the double centralizer theorem).

Finally, we will discuss what is known about Morita equivalence between the specializations $\epsilon H_{t,c} \epsilon$ and $H_{t,c}$ to numerical parameters.
8.1. Universal SRA as a universal deformation.

**Theorem 8.1.** Suppose that $V$ is symplectically irreducible. Then $H$ is a deformation of $S(V)\#\Gamma$, and there is an isomorphism $P_{un} \cong P$ making $H$ and $H_{an}$ equivalent in the sense explained above.

**Proof.** We will check that $H_{un}$ is given by the same generators and relations as $H$.

Let us deal with generators first. Let $\pi$ denote the natural projection $H_{un} \to SV\#\Gamma$. Since $P_{un}$ has degree 2, $\pi$ identifies the degree 0 component of $H_{un}$ with $(S(V)\#\Gamma)^0 = \mathbb{C}\Gamma$ and the degree 1 component with $(S(V)\#\Gamma)^1 = V \otimes \mathbb{C}\Gamma$. In particular, there is a natural inclusion of $V$ into $H_{un}$. The $S(P_{un})$-subalgebra generated by $V$ and $\mathbb{C}\Gamma$ is graded and surjects onto $S(V)\#\Gamma$. It follows from the next exercise that this subalgebra coincides with $H_{un}$.

**Exercise 8.1.** Let $M = \bigoplus_{i=0}^{\infty} M$ be a graded module over $S(P)$, where $P$ is a vector space. Let $M_0$ be a graded $S(P)$-submodule such that $M_0 \to M/PM$. Show that $M_0 = M$.

So we get a natural epimorphism $S(P_{un}) \otimes T(V)\#\Gamma \to H_{un}$.

Let us proceed to relations, i.e., to describing the kernel of the epimorphism above. For $u, v \in V \subset H_{un}$, the element $[u, v]$ has degree 2 and lies in ker $\pi$. But the degree 2 component of ker $\pi$ is $P_{un} \otimes \mathbb{C}\Gamma$. So there is a map $\kappa : A^2 V \to P_{un} \otimes \mathbb{C}\Gamma$ such that $[u, v] = \kappa(u, v)$.

Let $\tilde{H}_{un} := S(P_{un}) \otimes T(V)\#\Gamma/(u \otimes v - v \otimes u - \kappa(u, v))$. Then $\tilde{H}_{un}/P_{un} \tilde{H}_{un} = S(V)\#\Gamma$, while there is an epimorphism $\tilde{H}_{un} \to H_{un}$. Since $H_{un}$ is a free graded $S(P_{un})$-module, the following exercise implies $\tilde{H}_{un} = H_{un}$.

**Exercise 8.2.** Let $M_1, M_2$ be two nonnegatively graded $S(P)$-modules, where $P$ is a vector space. Suppose that $M_2$ is a graded free module. Consider an epimorphism $M_1 \to M_2$ that induces an isomorphism $M_1/PM_1 \cong M_2/PM_2$. Show that this epimorphism is an isomorphism.

We claim that there are $t', c_1', \ldots, c_m' \in P_{un}$ such that $\kappa(u, v) = t' \omega(u, v) + \sum_{i=1}^{m} c_i' \sum_{s \in S} \omega_s(u, v) s$. This follows from our computations in Lecture 6 (for example, using passing to a numerical specialization of $H_{un}$). Also we remark that $t', c_1', \ldots, c_m'$ is a basis in $P_{un}$ – here finally we will use that $H_{un}$ is a universal deformation, the arguments above worked for any deformation.

It is enough to show that $t', c_1', \ldots, c_m'$ span $P_{un}$ because $\dim P_{un} = m + 1$. Let $P_{un}'$ be the subspace spanned by $t', c_1', \ldots, c_m'$. Deformations $S(P_{un}) \otimes_{S(P_{un})} H_{un}$ and $H_{un}$ are equivalent for any linear map $P_{un} \to P_{un}'$ that is the identity on $P_{un}'$ (they are just algebras given by exactly the same relations). But a linear map $P_{un} \to P_{un}$ with $S(P_{un}) \otimes_{S(P_{un})} H_{un}$ and $H_{un}$ has to be unique and hence $P_{un}' = P_{un}$.

**Exercise 8.3.** Use the theorem to deduce that $H$ is a graded deformation of $S(V)\#\Gamma$ even if $V$ is not symplectically irreducible.

We remark that any graded deformation $H'$ of $S(V)\#\Gamma$ over $S(P')$ has no nontrivial self-equivalences. This is because any such self-equivalence is forced to be the identity on $\mathbb{C}\Gamma$ and $V$ (thanks to deg $P' = 2$). But the $(P')$-algebra $H'$ is generated by $\mathbb{C}\Gamma$ and $V$. So the equivalence is the identity and $H_{un}$ is a universal object in the categorical sense.

8.2. Algebra $eHe$ and bimodule $eHe$. Let $e$ be the idempotent $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}\Gamma$. We can view $e$ as an element of $H$ or of its numerical specialization $H_{t,e}$. Then we get spherical subalgebras $eHe \subset H, eH_{t,c}e \subset H_{t,e}$.

**Exercise 8.4.** Prove that $eHe$ is a graded deformation of $S(V)\Gamma$ over $S(P)$. Also prove that the specialization of $eHe$ at $t, c_1, \ldots, c_m \in \mathbb{C}$ coincides with $eH_{t,c}e$ so that $\text{gr} eH_{t,c}e = S(V)\Gamma$. 

We can consider the \( S(P) \)-module \( He \) and also its specializations \( H_{t,c}e \). The space \( He \) has commuting actions of \( H \) on the left and \( eHe \) on the right and so becomes an \( H-eHe \)-bimodule.

**Lemma 8.2.** The right \( eHe \)-module \( He \) is finitely generated.

**Proof.** We know that \( S(V) = He/PHe \) is finitely generated over \( S(V)^{\Gamma} = eHe/PeHe \). We can choose finitely many homogeneous generators \( m_1, \ldots, m_n \). Then lift them to homogeneous elements \( \tilde{m}_1, \ldots, \tilde{m}_n \) of \( He \). Exercise 8.1 implies that \( \tilde{m}_1, \ldots, \tilde{m}_n \) generate the right \( eHe \)-module \( He \). \( \square \)

Similarly, \( eH \) is a finitely generated left \( eHe \)-module.

**8.3. Double centralizer property.** We are going to prove that the algebras \( H_{t,c}, eH_{t,c}e \) are mutual centralizers in the bimodule \( H_{t,c}e \). One statement here is easy.

**Exercise 8.5.** The homomorphism \( eH_{t,c}e^{\text{opp}} \to \text{End}_{H_{t,c}e}(H_{t,c}e) \) is an isomorphism.

The following theorem is due to Etingof and Ginzburg, \([EG]\).

**Theorem 8.3.** The homomorphism \( H_{t,c} \to \text{End}_{eH_{t,c}e^{\text{opp}}}(H_{t,c}e) \) is an isomorphism.

**Proof.** The proof is organized as follows. We start with the case \( t = 0, c = 0 \). We first prove the injectivity, which is easier, and then the surjectivity, which is harder. After that the general case is done basically by passing to associated graded.

**Step 1.** We claim that the natural map \( S(V)^{\#\Gamma} \to \text{End}_{S(V)^{\Gamma}}(S(V)) \) is injective. In what follows we will identify \( S(V) \) with \( \mathbb{C}[V] \) using the identification of \( V \) and \( V^* \) coming from the symplectic form.

Let \( V^0 \) denote the subset in \( V \) consisting of all points with trivial stabilizers. This subset is open and, since \( \Gamma \) acts faithfully – only the unit element acts as \( 1 \) on \( V \), we have \( V^0 \neq \emptyset \). Let \( \sum_{\gamma} f_{\gamma} \gamma \) lie in the kernel of \( \mathbb{C}[V]^{\#\Gamma} \to \text{End}_{\mathbb{C}[V^{\Gamma}]}(\mathbb{C}[V]) \). This means that \( \sum_{\gamma \in \Gamma} f_{\gamma}\gamma(g) = 0 \) for any \( g \in \mathbb{C}[V] \). Pick \( v \in V^0 \). For any complex numbers \( z_{\gamma} \) there is \( g \in \mathbb{C}[V] \) such that \( g(\gamma^{-1}v) = z_{\gamma} \). It follows that \( \sum_{\gamma} f_{\gamma}(v)z_{\gamma} = 0 \) and so \( f_{\gamma}(v) = 0 \). Since \( V^0 \) is open and non-empty, we deduce that \( f_{\gamma} = 0 \).

**Step 2.** To prove that the homomorphism \( \mathbb{C}[V]^{\#\Gamma} \to \text{End}_{\mathbb{C}[V^{\Gamma}]}(\mathbb{C}[V]) \) is surjective we will need the following lemma.

**Lemma 8.4.** Let \( X \) be a smooth affine variety equipped with a free action of a finite group \( \Gamma \). Then the homomorphism \( \mathbb{C}[X]^{\#\Gamma} \to \text{End}_{\mathbb{C}[X^{\Gamma}]}(\mathbb{C}[X]) \) is an isomorphism.

**Proof of Lemma.** We remark that both \( \mathbb{C}[X]^{\#\Gamma} \) and \( \text{End}_{\mathbb{C}[X^{\Gamma}]}(\mathbb{C}[X]) \) are locally free \( \mathbb{C}[X]^{\#\Gamma} \)-modules of rank \( |\Gamma|^2 \). To prove this we only need to check that \( \mathbb{C}[X] \) is a locally free \( \mathbb{C}[X]^{\#\Gamma} \)-module of rank \( |\Gamma| \). Pick a point \( x \in X \) and let \( \pi : X \to X/\Gamma \) denote the quotient morphism. Then there are \( g_\gamma \in \mathbb{C}[X], \gamma \in \Gamma \), such that the matrix \( (g_\gamma(\gamma')x)_{\gamma,\gamma' \in \Gamma} \) is non-degenerate. These elements form a basis of the \( \mathbb{C}[X]^{\#\Gamma} \)-module \( \mathbb{C}[X] \) after an appropriate localization. This implies that \( \mathbb{C}[X] \) is locally free of rank \( \mathbb{C}[X]^{\#\Gamma} \).

Now to show that the homomorphism \( \mathbb{C}[X]^{\#\Gamma} \to \text{End}_{\mathbb{C}[X^{\Gamma}]}(\mathbb{C}[X]) \) is bijective it is enough to show that it is injective fiberwise, i.e., the induced homomorphism \( A_{y}^{\#\Gamma} \to \text{End}(A_{y}) \) is injective for any \( y \in X/\Gamma \), where \( A_{y} = \mathbb{C}[X]/\mathbb{C}[X]m_{y}, m_{y} \) being the maximal ideal of \( y \) in \( \mathbb{C}[X]^{\#\Gamma} \). But the algebra \( A_{y} \) is just the algebra of functions on \( \pi^{-1}(y) \), a free \( \Gamma \)-orbit. The injectivity is checked as in the proof of step 1. \( \square \)
Recall that $\Gamma \subset \text{Sp}(V)$. In particular, for any $\gamma \in \Gamma$ the fixed point subspace $V^\gamma$ has codimension at least 2. So $\text{codim}_V V \setminus V^0 \geq 2$. For every point $v \in V^0$ we can find $f_v \in \mathbb{C}[V]^\Gamma$ such that $f_v(v) \neq 0$, $f_v(V \setminus V^0) = 0$. Let $V^0 := \{ u \in V | f_u(u) \neq 0 \}$, this is a $\Gamma$-stable affine open subset of $V^0$. For convenience, we can choose a finite covering $V^0 = \bigcup_i V_i$ by subsets of the form $V^0$, let $f_i$ denote the corresponding polynomial, so that $\mathbb{C}[V_i] = \mathbb{C}[V]^f$ and $\mathbb{C}[V_i]^f = \mathbb{C}[V]^f$. By general Commutative algebra, $\text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_i])$ is just the localization of $\text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V])$ by $f_i$. In particular, we have a homomorphism $\iota_i : \text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V]) \to \text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_i])$. It is injective: if we have $\iota_i(\varphi)(f/f^k) = 0$, then $\iota_i(\varphi)(f) = 0$ for any $f \in \mathbb{C}[V]$. Pick $\varphi \in \text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V])$. Now, by Lemma applied to $X \equiv V_i$, we have $\iota_i(\varphi) = \sum_{\gamma \in \Gamma} f_i \gamma$ for some (uniquely determined) elements $f_i \gamma \in \mathbb{C}[V_i]$.

Set $V_{ij} = V_i \cap V_j$. We claim that $f_i|_{V_{ij}} = f_j|_{V_{ij}}$. We have, again injective, homomorphisms $\text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V]),\text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_i]),\text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_j]) \to \text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_{ij}])$, denote them by $\iota_{ij}, \iota'_j, \iota'_i$, respectively. Of course, $\iota_{ij} = \iota'_i \circ \iota_j = \iota'_j \circ \iota_i$. Clearly, $\iota'_i$ sends $\sum_{\gamma \in \Gamma} f_i^{\gamma} \gamma$ to the same element (where we now view the $f_i^{\gamma}$’s as elements of $\mathbb{C}[V_{ij}]$ not of $\mathbb{C}[V_i]$) and the similar claim holds for $\iota'_j$. But, by the lemma above, the natural homomorphism $\mathbb{C}[V_{ij}]^\# \Gamma \to \text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_{ij}])$ is injective. It follows that $f_i^{\gamma} = f_j^{\gamma}$ in $\mathbb{C}[V_i]$.

So the functions $f_i \gamma$ glue to a regular function $f_i$ on $V^0$. But recall that $\text{codim} V \setminus V^0 \geq 2$. It follows that $f_i$ is regular on the whole $V$. The element $\sum_{\gamma} f_i \gamma \in \mathbb{C}[V]^\# \Gamma$ produces the endomorphism $\varphi$. This follows, for example, from the injectivity of $\text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V]) \to \text{End}_{\mathbb{C}[V]^f}(\mathbb{C}[V_i])$ and the construction of $f_i$.

**Step 3.** Let us equip the algebra $\text{End}_{eH\text{t}_e,\text{opp}}(H_t,e)$ with a filtration.

**Exercise 8.6.** Let $A$ be a $\mathbb{Z}_{\geq 0}$-filtered algebra and $M$ be its module. Equip $M$ with a filtration compatible with that on $A$ in such a way that $\text{gr} M$ is finitely generated $\text{gr} A$-module. We set $\text{End}_A(M)^{\leq n} := \{ \psi \in \text{End}_A(M)|\psi(M^{\leq n}) \subset M^{\leq n+m}, \forall m \}$.

1. Show that this is a $\mathbb{Z}$-filtration and that $\text{End}_A(M)^{\leq n} = 0$ for $n < 0$.
2. Construct a natural homomorphism $\text{gr} \text{End}_A(M) \to \text{End}_{\text{gr} A}(\text{gr} M)$ of graded algebras.

3. Show that this homomorphism is injective.

**Exercise 8.7.** Let us retain the conventions of the previous exercise. Let $B$ be another $\mathbb{Z}_{\geq 0}$-filtered algebra such that $M$ becomes a filtered $A \otimes B$-module. Show that there is a homomorphism $\text{End}_B(M) \to \text{End}_A(M)$ of filtered algebras. Moreover, show that the composite homomorphism $\text{gr} B \to \text{gr} \text{End}_A(M) \to \text{End}_{\text{gr} A}(\text{gr} M)$ coincides with the homomorphism induced by the action of $\text{gr} A \otimes \text{gr} B$ on $\text{gr} M$.

The right $eH\text{t}_e,\text{opp}$-module $H_t,e$ satisfies the conditions of the exercise. So we get a monomorphism $\text{gr} \text{End}_{eH\text{t}_e,\text{opp}}(H_t,e) \to \text{End}_{\text{gr} eH\text{t}_e,\text{opp}}(\text{gr} H_t,e) = \text{End}_{S(V)^\# \Gamma}(S(V))$ of graded algebras. Clearly, $H_t,e$ is filtered as an $H_t,e \otimes eH\text{t}_e,\text{opp}$-module. So we get the induced homomorphism $S(V)^\# \Gamma = \text{gr} H_t,e \to \text{gr} \text{End}_{eH\text{t}_e,\text{opp}}(H_t,e)$. The composite homomorphism $S(V)^\# \Gamma \to \text{End}_{S(V)^\# \Gamma}(S(V))$ is the same as one from Steps 1,2 and so is an isomorphism. We deduce that $\text{gr} H_t,e \to \text{gr} \text{End}_{eH\text{t}_e,\text{opp}}(H_t,e)$ is an isomorphism. According to the following exercise, the homomorphism $H_t,e \to \text{End}_{eH\text{t}_e,\text{opp}}(H_t,e)$ is an isomorphism.

**Exercise 8.8.** Let $M,N$ be $\mathbb{Z}$-filtered vector spaces such that $M^{\leq n} = 0$ for $n < 0$. Let $\varphi : M \to N$ be a filtration preserving linear map. Show that if $\text{gr} \varphi : \text{gr} M \to \text{gr} N$ is an isomorphism, then $\varphi$ is an isomorphism.
Problem 8.9. Show that $H,eHe$ also satisfy the double centralizer property.

8.4. Spherical parameters. The double centralizer property for $eH_{t,c}e$ and $H_{t,c}$ is not to be confused with the Morita equivalence condition: $H_{t,c}eH_{t,c} = H_{t,c}$, the latter is far more restrictive. For example, the irreducible modules for $S(V)\#\Gamma$ with zero action of $V$ are precisely the $\Gamma$-irreducibles. All such modules but the trivial one are annihilated by $e$.

The parameter $(t,c)$ such that $H_{t,c}eH_{t,c} = H_{t,c}$ is called spherical. Let us explain what is known about spherical parameters when $t = 1$. The case when $t = 0$ will be mentioned in the next lecture. This dichotomy is justified by the next exercise.

Exercise 8.10. Let $a \in \mathbb{C}^\times$. Establish a natural isomorphism between $H_{t,c}$ and $H_{at,ac}$.

In the case when $\dim V = 2$ the description of spherical parameters was obtained by Crawley-Boevey and Holland in [CBH]. Namely, recall that (in the notation of lectures 1-4) to $t, c$ we can assign the $r + 1$-tuple $(\lambda_i)_{i=0}^r$ by

$$\lambda_i = \text{tr}_{N_i}(t\omega(u,v) + \sum_{i=1}^m c_i \sum_{s \in S_i} \omega_s(u,v))$$

Then the parameter $(t,c)$ is spherical (no matter whether $t = 0$ or not) if and only if $\sum_{i=1}^r \lambda_i \alpha_i \neq 0$ for any root $\alpha = \sum_{i=1}^n \alpha_i \epsilon_i$ of the corresponding finite root system (we use the convention that $\epsilon_1, \ldots, \epsilon_r$ correspond to simple roots).

The answer is known (and easy to state) also for the Rational Cherednik algebra of type $A$ – corresponding to the group $S_n$ and the double of its reflection representation. In this case, $c$ is a single complex number. It is known, see [BEG], that $(1,c)$ is spherical if and only if $c \neq \frac{r}{d}$, where $d = 2, 3, \ldots, n$ and $r$ is an integer with $-d < r < 0$.

Dunkl and Griffeth, [DG], obtained the description of the spherical parameters for the complex reflection groups $G(\ell,1,n)$. The answer is too complicated to be reproduced here.

Finally, let us mention that there is a conjecture of Etingof on the structure of the spherical parameters for all groups of the form $\Gamma_n = S_n \times \Gamma_1^n$, [E], that generalizes results of Dunkl and Griffeth. At the moment it is unclear how to prove that conjecture.

References


