LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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7. Hochschild cohomology and deformations

In the previous lecture we have introduced Hochschild cohomology. These were cohomology of the complex of the spaces $C^n(A,M)=\mathrm{Hom}_{\mathbb{C}}(A^{\otimes n},M)$ with differentials

$$\begin{aligned}df(a_1\otimes a_2\otimes\ldots\otimes a_{n+1})&=f(a_1\otimes\ldots\otimes a_n)a_{n+1}-f(a_1\otimes\ldots\otimes a_{n-1}\otimes a_na_{n+1})+\\&+f(a_1\otimes\ldots\otimes a_{n-1}a_n\otimes a_{n+1})-\ldots+(-1)^nf(a_1a_2\otimes a_3\ldots\otimes a_{n+1})+\\&+(-1)^{n+1}a_1f(a_2\otimes a_3\otimes\ldots\otimes a_{n+1}).\end{aligned}$$

We also have seen that $\mathrm{HH}^n(A,M)=\mathrm{Ext}^n(A,M)$.

Today we will see that that if A_0 is a $\mathbb{Z}_{\geqslant 0}$-graded algebra, then the space $\mathrm{HH}^n(A_0):=\mathrm{HH}^n(A_0,A_0)$ can be basically $\mathbb{Z}$-graded. Then we will see that the graded *1st order* deformations are described by the group $\mathrm{HH}^2(A_0)^{-2}$. Next, we will show that if $\mathrm{HH}^3(A_0)^i=0$ for all $i\leqslant -4$, then all deformations are *unobstructed*. From here, modulo $\mathrm{HH}^2(A_0)^i=0$ for $i<-2$, we will deduce the existence of some *universal* graded deformation of A over the algebra $S\left((\mathrm{HH}^2(A_0)^{-2})^*\right)$. After that we will compute the relevant cohomology groups for $S(V)\#\Gamma$, where Γ is a finite subgroup of $\mathrm{Sp}(V)$. We will see that $\mathrm{HH}^3(S(V)\#\Gamma)^i=0$ for $i\leqslant -4$ and $\mathrm{HH}^3(S(V)\#\Gamma)^i=0$ for $i\leqslant -3$, while $\mathrm{HH}^2(S(V)\#\Gamma)^{-2}$ has dimension $m+1$, where m is the number of conjugacy classes of symplectic reflections in Γ, and $\mathrm{HH}^2(S(V)\#\Gamma)^i=0$ for $i<-2$. Then to prove that the universal SRA H is the universal deformation will be relatively easy. This will be done in the next lecture.

7.1. **Hochschild cohomology of graded algebras.** Let A be a $\mathbb{Z}$-graded algebra and M be a $\mathbb{Z}$-graded A-bimodule (meaning that the structure map $A\otimes M\otimes A\to M$ preserves the gradings, where we set $(A\otimes M\otimes A)^n:=\bigoplus_{i+j+k=m}A^i\otimes M^j\otimes A^k$). The space $A^{\otimes n}$ is also naturally graded. Set

$$C^n(A,M)^m:=\{f:A^{\otimes n}\to M|f\left((A^{\otimes n})^i\right)\subset M^{i+m},\forall i\}.$$

The formula for the differential implies that $d(C^n(A,M)^m)\subset C^{n+1}(A,M)^m$. Of course, in general, $C^n(A,M)$ does not need to coincide $\bigoplus_{m\in\mathbb{Z}}C^n(A,M)^m$ (basically, the former space is very large if A is infinite dimensional; it looks rather like the direct product of its graded components). We will modify the definition of $C^n(A,M)$ so that it coincides with $\bigoplus_{m\in\mathbb{Z}}C^n(A,M)^m$ and also modify the definition of $\mathrm{HH}^n(A,M)$ accordingly. The coincidence with the Ext groups is still true, if one modifies the definition of the Ext's in a similar way (considering graded resolutions and taking only homomorphisms sitting in finite number of degrees, the standard resolution of A is definitely graded).

7.2. $\mathrm{HH}^2(A)$ **and 1st order deformations.** Pick a finite dimensional vector space P and consider the graded algebra $S(P)/(P^2)=\mathbb{C}\oplus P$. Let A_0 be a $\mathbb{Z}_{\geqslant 0}$-graded algebra. By a 1st order (graded) deformation of A we mean a graded deformation of A_0 over $S(P)/(P^2)$. If A_1 is such a deformation, then it has the ideal $P\otimes A_0$ that squares to 0 and $A_1/P\otimes A_0=$

A_0. Definitely, if A is a graded deformation of A_0 over $S(P)$, then A/AP^2 is a 1st order deformation of A_0. So understanding 1st order deformation is the 1st step in understanding deformations over $S(P)$.

In what follows we will always assume that the degree of P is 2.

We denote the product on A_0 as usual, ab. We also use the same notation for the maps $A_0 \otimes (P \otimes A_0), (P \otimes A_0) \otimes A_0 \to P \otimes A_0$ induced by the product on A_0. Let μ^1 denote the product on A_1. We can represent A_1 as a direct sum of vector spaces $A_0 \oplus P \otimes A_0$ if we choose a graded section of the projection $A_1 \twoheadrightarrow A_0$. The values $\mu^1(a, b)$ are determined uniquely from μ_0 if one of the arguments a, b lies in $P \otimes A_0$: it is zero if both lie in $P \otimes A_0$. Also $\mu^1(a, p \otimes b') = p\mu^1(a, b') = p \otimes ab$ because μ^1 coincides with the product on A_0 modulo P. So μ^1 is recovered from its values on $A_0 \otimes A_0$. Let us write $\mu^1(a, b) = \mu_0(a, b) + \mu_1(a, b)$, where $\mu_1 : A_0 \otimes A_0 \to P$ (that depends on the choice of σ). In other words, $\mu_1 \in P \otimes C^2(A_0, A_0)$.

We want A_1 to be associative. As the following exercise shows, this is equivalent to $d\mu_1 = 0$, where d is the Hochschild differential.

Exercise 7.1. *Show that* $\mu^1(\mu^1(a, b), c) = \mu^1(a, \mu^1(b, c))$ *for* $a, b, c \in A_0$ *is equivalent to* $\mu_1(a, b)c + \mu_1(ab, c) = a\mu_1(b, c) + \mu_1(a, bc)$, *i.e., to* $d\mu_1 = 0$.

Let $Z^2(A_0)$ denote the space of Hochschild 2-cocycles. We have $\mu_1 \in P \otimes Z^2(A_0)$. But also the algebra A_1 has to be graded: $\mu^1(A_1^i, A_1^j) \subset A_0^{i+j}$ or equivalently, $\mu_1(A_0^i, A_0^j) \subset (P \otimes A_0)^{i+j} = P \otimes A_0^{i+j-2}$. So $\mu_1 \in P \otimes Z^2(A_0)^{-2}$.

It is natural to identify certain graded deformations. Namely, we say that graded $S(P)/(P^2)$-algebras A_1, A_1' are equivalent if there is an isomorphism $\sigma : A_1 \to A_1'$ of graded $S(P)/(P^2)$-algebras that is the identity modulo P. If we view A_1, A_1' as the space $A_0 \oplus P \otimes A_0$ with products μ, μ', then σ is the identity on $P \otimes A_0$ and on A_0 is given by $a \mapsto a + \sigma_1(a)$, where $\sigma_1 \in P \otimes C^1(A_0)$. Of course, this isomorphism preserves the gradings if and only if $\sigma_1 \in P \otimes C^1(A_0)^{-2}$.

Exercise 7.2. *The equality* $\sigma \circ \mu'^1 = \mu^1 \circ \sigma$ *is equivalent to* $\mu_1'(a, b) + \sigma(ab) = \mu_1(a, b) + a\sigma(b) + \sigma(a)b$, *i.e., to* $\mu' = \mu + d\sigma$.

So the classes in $Z^2(A_0)^{-2}$ correspond to equivalent deformations if and only if they are cohomologous. In other words, the equivalence classes of the 1st order graded deformations over $S(P)/(P^2)$ are parameterized by $P \otimes \mathrm{HH}^2(A_0)^{-2}$.

Now suppose that $\mathrm{HH}^2(A_0)^{-2}$ is finite dimensional. Then $P \otimes \mathrm{HH}^2(A_0)^{-2} = \mathrm{Hom}(\mathrm{HH}^2(A_0)^{-2*}, P)$. We can consider the space $P_{un} := \mathrm{HH}^2(A_0)^{-2*}$. Take the deformation $A_{1,un}$ over $S(P_{un})/(P_{un}^2)$ corresponding to $1 \in \mathrm{Hom}(\mathrm{HH}^2(A_0)^{-2*}, P_{un})$. By construction, this deformation has the following universal property: for any other 1st order graded deformation A_1 over $S(P)/(P^2)$, there is a unique linear map $P_{un} \to P$ such that $A_1 \sim S(P) \otimes_{S(P_{un})} A_{1,un}$.

7.3. $\mathrm{HH}^3(A_0)$ **and obstructions.** Of course, a graded deformation of A_0 over $S(P)$ also produces graded deformations over all $S(P)/(P^k)$. So we can try to produce a deformation step by step, lifting a deformation A_{k-1} over $S(P)/(P^k)$ (with $k \geqslant 2$) to a deformation A_k over $S(P)/(P^{k+1})$. The kernel of $A_k \twoheadrightarrow A_{k-1}$ is the ideal $S^k(P) \otimes A_0$ annihilated by the multiplication by P. We can decompose A_{k-1} and A_k as

$$A_{k-1} = A_0 \oplus P \otimes A_0 \oplus \ldots \oplus S^{k-1}(P) \otimes A_0, \quad A_k = A_0 \oplus P \otimes A_0 \oplus \ldots \oplus S^k(P) \otimes A_0.$$

Due to $S(P)$-linearity, the products μ^{k-1} on A_{k-1} and μ^k on A_k are recovered from their restrictions to $A_0 \otimes A_0$. There we can represent them in the form $a \cdot b = \sum_{i=0}^{k-1} \mu_i(a, b)$ in

A_{k-1} and $a \cdot b = \sum_{i=0}^{k} \mu_i(a,b)$ in A_k. Here μ_i is a graded map $A_0 \otimes A_0 \to S^i(P) \otimes A_0$, we know $\mu_0, \ldots, \mu_{k-1}$ and want to determine all possible μ_k.

The product on A_k has to be associative. We only need to check that the terms of $(a \cdot b) \cdot c$ and of $a \cdot (b \cdot c)$ that lie in $S^k(P) \otimes A$ coincides. We arrive at the equality

$$\sum_{i=0}^{k} \mu_i(\mu_{k-i}(a,b),c) = \sum_{i=0}^{k} \mu_i(a, \mu_{k-i}(b,c))$$

Let us move all terms containing μ_k to the left and all other (known) terms to the right. We get

$$(1) \quad \mu_k(a,b)c + \mu_k(ab,c) - \mu_k(a,bc) - a\mu_k(b,c) = \sum_{i=1}^{k-1} \left(\mu_i(a, \mu_{k-i}(b,c)) - \mu_i(\mu_{k-i}(a,b),c)\right).$$

The left hand side is $d\mu_k(a,b,c)$. It turns out that the right hand side is always a cocycle. This is a quite unpleasant computation, where one needs to use the fact that the products $\bigoplus_{i=0}^{j} \mu_j(a,b)$ are associative for all $j < k$ (and so $d\mu_j$ equals to the expression similar to the r.h.s. of (1)).

Exercise 7.3. *Show that the r.h.s. of (1) is a cocycle when $k = 2$.*

Problem 7.4. *Show that the r.h.s. of (1) is a cocycle for general k.*

In order for μ_k to exist the r.h.s., that belongs to $C^3(A,A)^{-2k}$, has to be a coboundary. This is automatically true when $\mathrm{HH}^3(A)^{-2k} = 0$.

Now suppose that, for whatever reasons, μ_k exists. Then it is defined up to adding $\nu \in C^2(A,A)^{-2k}$. As in the case of $k = 1$, we can define an equivalence relation on extensions of A_{k-1}: A_k and A'_k are equivalent if there is a graded $S(P)$-linear algebra isomorphism $\sigma : A_k \to A'_k$ that is the identity on A_{k-1}. Such an isomorphism is given by $\sum_{i=0}^{k} a_i \mapsto \sum_{i=0}^{k} a_i + \sigma_k(a_0)$, where $a_i \in S^i(P) \otimes A_0$ and $\sigma_k \in S^k(P) \otimes C^1(A_0, A_0)^{-2k}$. Similarly to the above, the condition that σ is an algebra isomorphism is equivalent to $\mu'_k - \mu_k = d\sigma_k$. We conclude that the equivalence classes of extensions A_k of A_{k-1} are parameterized by $\mathrm{HH}^2(A_0)^{-2k}$. In particular, if $\mathrm{HH}^2(A_0)^{-2k} = 0$, such extension is unique (and it exists provided $\mathrm{HH}^3(A_0)^{-2k} = 0$).

Finally, let us explain how to get a graded deformation A out of consecutive extensions $\ldots \twoheadrightarrow A_k \twoheadrightarrow A_{k-1} \twoheadrightarrow \ldots \twoheadrightarrow A_0$. The first idea is perhaps to take the inverse limit, but this is not what we want as the inverse limit has no grading (take the algebra of formal power series, for example). The correct answer is that one needs to take the inverse limit component wise. Namely, we have the inverse system (A_k^i) for each $i \geqslant 0$. We remark that this system stabilizers: the natural epimorphism $A_{k+1}^i \twoheadrightarrow A_k^i$ is an isomorphism for $2k > i$. This is because the kernel of $A_{k+1} \twoheadrightarrow A_k$ is generated by $S^{k+1}(P)$. Since the degrees for A are $\geqslant 0$ and the degree of $S^{k+1}(P)$ is $2(k+1)$, we see that the smallest degree in the kernel of $A_{k+1} \twoheadrightarrow A_k$ is $2(k+1)$, hence our claim. We set $A^i := \varprojlim_k A_k^i$. Then for A we take $\bigoplus_{i=0}^{+\infty} A$.

Exercise 7.5. *Equip A with a graded $S(P)$-algebra structure so that we have epimorphisms $A \twoheadrightarrow A_k$ for all k. Further, check that A is a free graded $S(P)$-module.*

7.4. **Universal deformation.** Let us assume that $\dim \mathrm{HH}^2(A_0)^{-2} < \infty, \mathrm{HH}^2(A_0)^i = 0$ for $i < -2$ and $\mathrm{HH}^3(A)^i = 0$ for $i < -3$. Consider the 1st order deformation $A_{1,un}$ constructed above. Thanks to our assumptions, we see by induction that there are uniquely determined (up to an equivalence) deformations $A_{un,k}$ such that $A_{un,k}$ lifts $A_{un,k-1}$. As explained above, we get a graded deformation A_{un} of A_0 over $S(P_{un}), P_{un} = \mathrm{HH}^2(A)^{-2*}$. Let A be a graded deformation of A_0 over $S(P)$.

Proposition 7.1. *There is a unique linear map $P_{un} \to P$ such that the deformations $S(P) \otimes_{S(P_{un})} A_{un}$ and A are equivalent (meaning that there is a graded $S(P)$-algebra isomorphism $S(P) \otimes_{S(P_{un})} A_{un} \xrightarrow{\sim} A$ that is the identity modulo P).*

Proof. Set $A' := S(P) \otimes_{S(P_{un})} A$. We will prove by induction on k that A'_k and A_k are equivalent, then it will imply that A' and A are so. We have already seen that there is a unique map $P_{un} \to P$ that makes A'_1 and A_1 equivalent. Since $\mathrm{HH}^2(A_0)^i = 0$ for $i < -2$, the equivalence class of A_k is determined by that of A_1. But as A_1 and A'_1 have the same equivalence classes, A'_k and A_k are equivalent. $\square$

Of course, the previous proposition is not yet a universality property, for that one wants an equivalence to be unique. The next exercise explains when this is so (in fact, in the case of $A_0 = S(V)\#\Gamma$ to see the uniqueness is easier).

Exercise 7.6. *Show that if $\mathrm{HH}^1(A_0)^i = 0$ for $i \leqslant -2$, then an equivalence in the proposition is unique.*

Problem 7.7. *Let A be a graded deformation of A_0 over $S(P)$. Describe the set of all auto-equivalences of A in terms of the groups $\mathrm{HH}^1(A_0)^i$ with $i \leqslant -2$.*

7.5. **Computations for $S(V)\#\Gamma$.** Now we set $A_0 = S(V)\#\Gamma$, where V is symplectically irreducible. We want to prove that $\mathrm{HH}^2(A_0)^i = 0$ for $i < -2$, that $\mathrm{HH}^3(A_0)^{-2} = 0$ for $i < -3$ and that $\dim \mathrm{HH}^2(A_0)^{-2} = m + 1$, where m is the number of classes of complex reflections. First, we are going to relate the computation of HH for $S(V)\#\Gamma$ for computations for $S(V)$.

Let B be an associative graded algebra acted on by Γ (so that Γ preserves the grading). Then we introduce a B-bimodule $B\gamma$ as follows. As a graded left B-module, $B\gamma = B$, but the right action is given by $(m\gamma)b = (m\gamma(b))\gamma$. We remark that Γ naturally acts on the direct sum $\bigoplus_{\gamma\in\Gamma} B\gamma$: $\gamma'(b\gamma) = \gamma'(b)\gamma'\gamma\gamma'^{-1}$. Of course, the B-bimodule $\bigoplus_{\gamma\in\Gamma} B\gamma$ is nothing else but $B\#\Gamma$, and the Γ-action on the direct sum is the adjoint Γ-action on $B\#\Gamma$. So we get a Γ action on $\bigoplus_{\gamma\in\Gamma} \mathrm{HH}^i(B, B\gamma)$ (induced by the Γ-actions on B and $\bigoplus_{\gamma\in\Gamma} B\gamma$) and it makes sense to speak about Γ-invariants.

Lemma 7.2. *We have an isomorphism $\mathrm{HH}^j(B\#\Gamma, B\#\Gamma) \cong \left(\bigoplus_{\gamma\in\Gamma} \mathrm{HH}^j(B, B\gamma)\right)^\Gamma$ of graded vector spaces.*

Proof. Let $B\text{-Bimod}^\Gamma$ denote the category of Γ-equivariant B-bimodules, i.e., B-bimodules M equipped with a Γ-action making the structure map $B \otimes M \otimes B \to M$ equivariant. We have a forgetful functor $B\#\Gamma\text{-Bimod} \to B\text{-Bimod}^\Gamma$ that remembers the adjoint action of Γ. This functor has left adjoint, $M \mapsto M\#\Gamma$, where, as a vector space, $M\#\Gamma = M\otimes\mathbb{C}\Gamma$, and the left and right multiplications by elements of $B\#\Gamma$ are defined similarly to the multiplication on $B\#\Gamma$ itself:

$$b \otimes \gamma \cdot m \otimes \gamma' = b\gamma(m) \otimes \gamma\gamma', \quad m \otimes \gamma' \cdot b \otimes \gamma = m\gamma'(b) \otimes \gamma'\gamma.$$

The functor is exact. For any $N \in B\#\Gamma\text{-Bimod}$ we have an isomorphism of functors $M \mapsto \mathrm{Hom}_{B\#\Gamma-Bimod}(M\#\Gamma, N)$ and $M \mapsto \mathrm{Hom}_{B-\mathrm{Bimod}^\Gamma}(M, N) = \mathrm{Hom}_{B-\mathrm{Bimod}}(M, N)^\Gamma$. Being the left adjoint functor of an exact functor, $\bullet\#\Gamma$ maps projective objects to projective objects. Any Γ-equivariant B-bimodule has a Γ-equivariant projective resolution (say, by free modules). By some basic Homological algebra, it follows that $\mathrm{Ext}^i_{B\#\Gamma-\mathrm{Bimod}}(M\#\Gamma, N)$ is naturally identified with $\mathrm{Ext}^i_{B-Bimod^\Gamma}(M,N)=\mathrm{Ext}^i_{B-Bimod}(M,N)^\Gamma$. Plugging $M=B, N=B\#\Gamma$, and using the coincidence of $\mathrm{HH}^\bullet$ with $\mathrm{Ext}^\bullet$, we complete the proof of the lemma. □

Now let $B=S(V)$. There is an advantage of the $S(V)$-bimodules $S(V)\gamma$ over the $S(V)\#\Gamma$-bimodule $S(V)\#\Gamma$: in a suitable basis $x_1,\ldots,x_n$ (depending on γ), the element γ is diagonalizable, $\gamma=\mathrm{diag}(\gamma_1,\ldots,\gamma_n)$. Then $S(V)\gamma=\bigotimes_{i=1}^n \mathbb{C}[x_i]\gamma_i$, where we view γ_i as an element of a suitable cyclic group acting on a 1-dimensional space.

Here is the following general fact from Homological algebra that is an analog of the Künneth formula from Topology:

$$\mathrm{Ext}^\bullet_{B_1\otimes B_2-Bimod}(B_1\otimes B_2, M_1\otimes M_2)=\mathrm{Ext}^\bullet_{B_1-Bimod}(B_1,M_1)\otimes \mathrm{Ext}^\bullet_{B_2-Bimod}(B_2,M_2),$$

where the homological degree on the l.h.s. corresponds to the sum of the homological degrees on the r.h.s. This isomorphism is compatible with gradings if the algebras and bimodules under consideration are graded. We deduce that

$$\mathrm{HH}^\bullet(S(V),S(V)g)=\bigotimes_{i=1}^n \mathrm{HH}^\bullet(\mathbb{C}[x_i],\mathbb{C}[x_i]\gamma_i).$$

So our next goal is to compute $\mathrm{HH}^\bullet(\mathbb{C}[x],\mathbb{C}[x]\gamma)$, where now γ is an element of some cyclic group acting on $\mathbb{C}[x]$ so that for $m\in C[x]\gamma$ we have $m\cdot x:=\gamma m x$. For a graded space M and $n\in\mathbb{Z}$ we write $M[n]$ for the same vector space but with shifted grading: $M[n]^m:=M^{n+m}$.

Lemma 7.3. (i) *We have* $\mathrm{HH}^i(\mathbb{C}[x],\mathbb{C}[x]\gamma)=0$ *for any* γ *and* $i\geqslant 2$.

(ii) *We have* $\mathrm{HH}^0(\mathbb{C}[x],\mathbb{C}[x])=\mathbb{C}[x]$ *with the usual grading, and* $\mathrm{HH}^1(\mathbb{C}[x],\mathbb{C}[x])=\mathbb{C}[x][1]$.

(iii) *Suppose* γ *is nontrivial. Then* $\mathrm{HH}^0(\mathbb{C}[x],\mathbb{C}[x]\gamma)=0$ *and* $\mathrm{HH}^1(\mathbb{C}[x],\mathbb{C}[x]\gamma)=\mathbb{C}[1]$.

Proof. An advantage of dealing with computing the Hochschild cohomology of the algebras $\mathbb{C}[X]$, where X is a smooth affine variety, is that we can take a Koszul type resolution of the $\mathbb{C}[X]$-bimodule $\mathbb{C}[X]$. In the case of $\mathbb{C}[x]$ this resolution is: $\mathbb{C}[x]\otimes\mathbb{C}[x]\to\mathbb{C}[x]\otimes\mathbb{C}[x]$, where the map given by $a\otimes b\to ax\otimes b-a\otimes xb$. Since the length of the resolution is 1, we immediately get (i).

Exercise 7.8. *Use the Koszul resolution to check that* $\mathrm{HH}^\bullet(\mathbb{C}[x],\mathbb{C}[x]\gamma)$ *is the cohomology of the complex* $\mathbb{C}[x]\gamma\to\mathbb{C}[x]\gamma$*, where the map is a left* $\mathbb{C}[x]$*-module homomorphism given by* $1\gamma\mapsto(x-\gamma(x))\gamma$*, where we shift the grading on the target module by* 1 *(so that* 1 *there has degree* -1*).*

So when $\gamma=1$, the map is 0 and we get (ii). For $\gamma\neq 1$, the map is injective, its cokernel is $\mathbb{C}$ sitting in degree -1. This is (iii). □

Now return to our situation, where we need to compute certain graded components of $\left(\bigoplus_{\gamma\in\Gamma}\mathrm{HH}^\bullet(S(V),S(V)\gamma)\right)^\Gamma$. We have

$$\mathrm{HH}^i(S(V),S(V)\gamma)=\bigoplus\bigotimes_k \mathrm{HH}^{i_k}(\mathbb{C}[x_k],\mathbb{C}[x_k]\gamma_k)^{j_k},$$

where the summation is taken over all n-tuples $(i_1,\ldots,i_n)$ of 0 and 1, and $(j_1,\ldots,j_n)$ of integers $\geqslant -1$ such that $i_1+\ldots+i_n=i, j_1+\ldots+j_n=j$. By the previous lemma we have $\bigotimes_k \mathrm{HH}^{i_k}(\mathbb{C}[x_k],\mathbb{C}[x_k]\gamma_k)^{j_k}=0$ if for $\gamma_k\neq 1$ we have $i_k\neq -1, j_j\neq -1$.

Exercise 7.9. *Show that $\gamma \in \Gamma$ has even number of eigenvalues different from 1. Deduce that $\mathrm{HH}^2(S(V), S(V)\gamma)^i = 0$ for $i < -2$ and $\mathrm{HH}^3(S(V), S(V)\gamma)^i = 0$ for $i < -3$.*

It remains to show that $\dim \left(\bigoplus_{\gamma\in\Gamma} \mathrm{HH}^\bullet(S(V), S(V)\gamma)^{-2}\right)^\Gamma = m+1$. This is achieved in the following exercise.

Exercise 7.10.
(1) *Suppose that γ is not a symplectic reflection. Prove that any element in $\mathrm{HH}^\bullet(S(V), S(V)\gamma)$ has homological degree at least 4.*
(2) *Let γ be a symplectic reflection. Show that $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2}$ is one dimensional.*
(3) *Let $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_m$. Show that $\dim \left(\bigoplus_{\gamma\in S_i} \mathrm{HH}^2(S(V), S(V)\gamma)^{-2}\right)^\Gamma = 1$.*
(4) *Show that, as a Γ-module, $\mathrm{HH}^2(S(V), S(V))^{-2}$ is $\bigwedge^2 V$. Deduce that*
$$\dim \left(\mathrm{HH}^2(S(V), S(V))^{-2}\right)^\Gamma = 1.$$