4. Deformed preprojective algebras, cont’d

4.1. Recap. Recall that in the previous lecture we have identified \( C(x, y) \# \Gamma \) with \( T_{\Gamma}(C^2 \otimes \Gamma) \), and \( \mathbb{C}Q \) with \( T_{\langle \mathbb{C}Q \rangle}(\mathbb{C}Q)^1 \). Also recall that \( f \mathbb{C}(x, y) \# \Gamma f \cong \mathbb{C}Q \), where \( f = \bigoplus_{i=0}^r f_i \in \Gamma = \bigoplus_{i=0}^r \text{End}(N_i) \) with \( f_i \) being a primitive idempotent in \( \text{End}(N_i) \). Under this identification, \( f_i \in \mathbb{C}Q \) becomes the path \( \epsilon_i \).

Further, to \( i \in Q_0 \) we have assigned an element \( [a^*, a]_i \in \epsilon_i(\mathbb{C}Q)^2 \epsilon_i \) by the formula

\[
[a^*, a]_i = \bigoplus_{a \in Q_1, t(a) = i} a^*a - \sum_{a \in Q_1, h(a) = i} aa^*.
\]

Also to \( c \in (\Gamma)^\Gamma \) we assign \( \lambda = (\lambda_i)_{i \in Q_0} \) by \( \lambda_i = \text{tr}_{N_i} c \). The main result we are going to prove is a theorem of Crawley-Boevey and Holland.

**Theorem 4.1.** The ideal \( f(xy - yx - c)_{C(x, y) \# \Gamma} f \) is generated by the elements \( [a^*, a]_i - \lambda_i \epsilon_i, i \in Q_0 \).

A key step in the proof is the following lemma again due to Crawley-Boevey and Holland.

**Lemma 4.2.** To each \( a \in Q_1 \) one can associate \( \eta_a \in \text{Hom}_\Gamma(N_{t(a)}, C^2 \otimes N_{h(a)}), \theta_a \in \text{Hom}_\Gamma(N_{h(a)}), C^2 \otimes N_{t(a)} \) that combine to form bases in the spaces \( \text{Hom}_\Gamma(N_i, C^2 \otimes N_j) \) are all \( i, j \) and satisfy

\[
\sum_{a \in Q_1, t(a) = i} (1_{C^2} \otimes \theta_a) \eta_a - \sum_{a \in Q_1, h(a) = i} (1_{C^2} \otimes \eta_a) \theta_a = \delta_i(\zeta \otimes 1_{N_i}),
\]

(the equality of maps \( N_i \rightarrow C^2 \otimes C^2 \otimes N_j \)) for all \( i \).

To prove the lemma we have introduced explicit mutually inverse isomorphisms of the spaces \( \text{Hom}_\Gamma(M, C^2 \otimes M') \), \( \text{Hom}_\Gamma(C^2 \otimes M, M') \). Namely, we map \( \psi \in \text{Hom}_\Gamma(M, C^2 \otimes M') \) to \( \psi^\omega := (\omega \otimes 1_{M'}) \circ (1_{C^2} \otimes \psi) \in \text{Hom}_\Gamma(C^2 \otimes M, M') \), and we map \( \varphi \in \text{Hom}_\Gamma(C^2 \otimes M, M') \) to \( (1_{C^2} \otimes \varphi) \circ (\zeta \otimes 1_{M}) \in \text{Hom}(M, C^2 \otimes M') \). Here \( \omega \) is the skew-symmetric form on \( C^2 \) given by \( \omega(y, x) = 1 = -\omega(x, y) \) (and viewed as a map \( C^2 \otimes C^2 \rightarrow \mathbb{C} \) and \( \zeta = x \otimes y - y \otimes x \) (viewed as a map \( \mathbb{C} \rightarrow C^2 \otimes C^2 \)).

4.2. Proof of the CBH lemma. From now on we concentrate on the non-cyclic case. A special feature of this case is that \( \Gamma \) is a tree.

The spaces \( \text{Hom}_\Gamma(N_i, C^2 \otimes N_j) \) when \( i = t(a), j = h(a) \) or vice versa are 1-dimensional. For a moment, choose arbitrary nonzero \( \eta_a, \theta_a \), they are defined up to a nonzero scalar multiple. Then \( \theta_a^\omega \eta_a \) is a nonzero endomorphism of \( N_{t(a)} \), while \( \eta_a^\omega \theta_a \) is a nonzero endomorphism of \( N_{h(a)} \). Multiplying \( \theta_a \) by a nonzero scalar \( k \), we also multiply those two endomorphisms by \( k \). We claim that there are nonzero scalars \( d_i, i \in Q_0 \), with the property that (after rescaling the \( \theta_i \)’s) we get

\[
\theta_a^\omega \eta_a = d_{h(a)}1_{N_i}, \eta_a^\omega \theta_a = -d_{i}1_{N_h(a)}.
\]
This is a consequence of Q being a tree. Namely, we fix all $\eta_a$ and some vertex $i$. Pick $d_i$. This fixes $\theta_a$ for all $a$ with $h(a) = i$ (from $\theta_a \eta_a = d_{h(a)} 1_{N_{h(a)}}$) or $t(a) = i$ (from $\eta_a^\vee \theta_a = -d_{t(a)} 1_{N_{h(a)}}$) and so also $d_i$, for all vertices $j$ connected to $i$ (for example, if $h(a) = i$, then $d_{t(a)}$ is determined from $\eta_a^\vee \theta_a = -d_{t(a)} 1_{N_{h(a)}}$, where we now know the left hand side). Then we proceed with $i$ replaced by one of these $j$’s. Since our graph is a tree, we see that every vertex appears only ones, and when our argument finishes, we get all $d_i$ and all $\theta_a$ fixed.

The map $\eta_a \theta_a^\vee : C^2 \otimes N_{h(a)} \to C^2 \otimes N_{i(a)}$ therefore equals $d_{h(a)} \pi_{N_{i(a)}}$, where $\pi_{N_{i(a)}}$ is the projection to the summand $N_{t(a)}$ in $C^2 \otimes N_{h(a)}$. Similarly, $\theta_a \eta_a^\vee = -d_{t(a)} \pi_{N_{h(a)}}$. The modules $N_{t(a)}$ with $h(a) = i$, and $N_{h(a)}$ with $t(a) = i$ is a complete list of the simple summands of $C^2 \otimes N_i$. So for $i \in Q_0$ we have

$$\sum_{a \in Q, h(a) = i} \eta_a \theta_a^\vee - \sum_{a \in Q, t(a) = i} \theta_a \eta_a^\vee = d_i 1_{C^2 \otimes N_i}. \tag{3}$$

We want to tensor the previous equation with $1_{C^2}$ (on the left) and compose with $\varpi \otimes 1$ (on the right) so that we get an equality of maps $N_i \to C^2 \otimes C^2 \otimes N_i$. We have $(1_{C^2} \otimes \theta_a \eta_a^\vee) \circ (\varpi \otimes 1) = (1_{C^2} \otimes \theta_a)(1_{C^2} \otimes \eta_a^\vee)(\varpi \otimes 1_N) = (1_{C^2} \otimes \theta_a)\eta_a$. So what we get is

$$\sum_{a \in Q, h(a) = i} (1_{C^2} \otimes \eta_a) \theta_a - \sum_{a \in Q, t(a) = i} (1_{C^2} \otimes \theta_a) \eta_a = d_i \varpi \otimes 1_{N_i}. \tag{4}$$

To show that we can take $-\delta_i$ for $d_i$, we compose both sides of the equality on the left with $\omega \otimes 1$ (to get a map $N_i \to N_i$). We have $(\omega \otimes 1_{N_i})(1_{C^2} \otimes \theta_a)\eta_a = \theta_a \eta_a = d_{h(a)} 1_{N_i}$, and similarly $(\omega \otimes 1_{N_i})(1_{C^2} \otimes \eta_a)\theta_a = -d_{t(a)} 1_{N_i}$. So on the left hand side we get $-\sum_j d_j N_i$, where we sum over all $j$ connected to $i$. On the right hand side we get $-\omega(\varpi d_i 1_{N_i}) = -2d_i 1_{N_i}$. So $2d_i - \sum_j d_j = 0$ for each $i \in Q_0$. This is a linear system whose matrix is precisely the Cartan matrix of the extended Dynkin diagram. The space of solutions of this equation is 1-dimensional and is generated by $\delta_i$. Rescaling the maps $\theta_a$, we achieve $d_i = -\delta_i$.

4.3 Proof of Theorem 4.1. We need to interpret (1) so that it becomes an equality in $CQ = T_{(CQ)^0}(CQ)^1 = fC(x, y)\# \Gamma f$.

Hom$_{\Gamma}(N_i, C^2 \otimes N_i)$ is just Hom$_{\Gamma}(C \Gamma f_i, C^2 \varotimes C \Gamma f_i) = f_i(C^2 \varotimes C \Gamma)f_i = \epsilon_i(CQ)^1 \epsilon_i$. We take $\eta_a$ for $a$, and $\theta_a$ for $a^*$. Then, similarly, Hom$_{\Gamma}(N_i, C^2 \otimes C^2 \otimes N_i) = Hom_{\Gamma}(C \Gamma f_i, C^2 \otimes C^2 \Gamma f_i) = f_i(C^2 \varotimes C^2 \varotimes C \Gamma)f_i = \epsilon_i(CQ)^2 \epsilon_i$, and $(1 \otimes \theta_a)\eta_a \in Hom_{\Gamma}(N_i, C^2 \otimes C^2 \otimes N_i)$ is nothing else but $a^* a$.

The elements $xy - yx$ and $c$ of $C(x, y)\# \Gamma$ are $\Gamma$-invariant and hence commute with $C \Gamma$ and in particular, with the idempotents $f_i$. Under our identifications, the element $(xy - yx)f_i = f_i(xy - yx)f_i$ is nothing else but $\varpi \otimes 1_{N_i}$. So the CBH Lemma just says that

$$\sum_{a \in Q, t(a) = i} a^* a - \sum_{a \in Q, h(a) = i} a a^* = \delta_i(xy - yx)f_i$$

We remark that $\delta_i c f_i$ is precisely $\lambda_i f_i$, so

$$\sum_{a \in Q, t(a) = i} a^* a - \sum_{a \in Q, h(a) = i} a a^* - \lambda_i f_i = \delta_i(xy - yx - c)f_i$$

Now we notice that the left hand sides of the previous equality all lie in $f(xy - yx - c)c(x, y)\# \Gamma f$. On the other hand, let us show that $f(xy - yx - c)c(x, y)\# \Gamma f$ coincides with the ideal in $CQ$ generated by $(xy - yx - c)f$. Recall that $C \Gamma / C \Gamma = \Gamma$ and so there are elements $r_j, s_j \in \Gamma$ with $\sum_j r_j f s_j = 1$. So $\sum_j r_j (xy - yx - c)f s_j = \sum_j (xy - yx - c)r_j f s_j = xy - yx - c$. 

So $xy - yx - c$ lies in the ideal of $\mathbb{C}(x,y)^\# \Gamma$ generated by $(xy - yx - c)f$. It follows that $(xy - yx - c)\mathbb{C}(x,y)^\# \Gamma \cap f(\mathbb{C}(x,y)^\# \Gamma)f = (xy - yx - c)f_{\mathbb{C}Q}$.

4.4. Remarks and ramifications.

4.4.1. Orientation. Formally, the algebra $\Pi^\lambda$ (and the map $\mu : \text{Rep}(Q, \delta) \to \mathfrak{gl}(\delta)$) depends on the orientation. Pick an arrow $b \in Q_1$ and switch its orientation, let $b_1$ denote the same arrow with the inverted orientation. Let $\Pi^\lambda, \mu'$ be constructed from this new orientation. The map $\epsilon_i \mapsto \epsilon_i, a \mapsto a, a^* \mapsto a^*, a \in Q_1 \setminus \{b\}, b \mapsto b_1, b^* \mapsto -b_1$ extends to an automorphism of $\mathbb{C}Q$ that gives rise to an algebra isomorphism $\Pi^\lambda \to \Pi^\lambda$. Also this automorphism gives rise to a linear $\text{GL}(\delta)$-equivariant automorphism of $\text{Rep}(Q, \delta)$ that intertwines the maps $\mu$ and $\mu'$. So our constructions do not depend on the choice of an orientation up to distinguished isomorphisms.

4.4.2. Identification of $\text{Rep}_r(\mathbb{C}[x,y]^\# \Gamma, \mathbb{C}\Gamma) // \text{GL}(\mathbb{C}\Gamma)^\Gamma$ with $\mathbb{C}^2/\Gamma$. We have identified

$$\text{Rep}_r(\mathbb{C}[x,y]^\# \Gamma, \mathbb{C}\Gamma) // \text{GL}(\mathbb{C}\Gamma)^\Gamma$$

(viewed as a set of isomorphism classes of semisimple representations) with $\mathbb{C}^2/\Gamma$. Let us show that this is an identification of algebraic varieties. Recall the spherical subalgebra $\mathbb{C}[x,y]^\Gamma \cong e(\mathbb{C}[x,y]^\# \Gamma)e \subset \mathbb{C}[x,y]^\# \Gamma$. An element $\varphi \in \text{Rep}_r(\mathbb{C}[x,y]^\# \Gamma, \mathbb{C}\Gamma)$ restricts to a representation of $e(\mathbb{C}[x,y]^\# \Gamma)e$ in $e\mathbb{C}\Gamma = \mathbb{C}$. The latter is nothing else but a point of $\mathbb{C}^2/\Gamma$ and so we get another map $\xi : \text{Rep}_r(\mathbb{C}[x,y]^\# \Gamma, \mathbb{C}\Gamma) \to \mathbb{C}^2/\Gamma$ that is clearly $\text{GL}(\mathbb{C}\Gamma)^\Gamma$-equivariant and so descends to $\text{Rep}_r(\mathbb{C}[x,y]^\# \Gamma, \mathbb{C}\Gamma) // \text{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathbb{C}^2/\Gamma$. It is clear from the construction that it coincides with our previous map, given by taking the central character. Our new map is a morphism of algebraic varieties: the pull-back $\xi^*(F) = F \in \mathbb{C}[x,y]^\Gamma$ evaluated on a representation $\varphi$ is just a matrix coefficient of $\varphi$ evaluated on $eFe$ (or $F$) and so $\xi^*(F)$ is a polynomial on $\text{Rep}_r(\mathbb{C}[x,y]^\# \Gamma, \mathbb{C}\Gamma)$.

4.4.3. Scheme structure on $\mu^{-1}(0)/G$. Before we have viewed $\mu^{-1}(0)/G$ as a variety, but in fact, it is an affine scheme with the following algebra of functions: $(\mathbb{C}[R]/\mathbb{C}[R]\mu^*(\mathfrak{g}))^G$, where we write $G$ for $\text{GL}(\delta), \mathfrak{g}$ for $\mathfrak{gl}(\delta)$ and $R$ for $\text{Rep}(Q, \delta)$. Here $\mu^* : \mathfrak{g} \to \mathbb{C}[R]$ is a pull-back map induced by $\mu : R \to \mathfrak{g}$ (we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace pairings on all $\mathfrak{gl}_n(\mathbb{C})$).

It turns out however, that even the subscheme $\mu^{-1}(0) \subset R$ (not only the quotient $\mu^{-1}(0)/G$) is reduced (and is a complete intersection). This follows from the claim that every irreducible component of $\mu^{-1}(0)$ contains a $G$-orbit with trivial stabilizer (and also from some standard properties of moment maps). The claim about the existence of an orbit follows from the representation theory of quivers.

Problem 4.1. Check the claims of the previous paragraph by hand in the case of the cyclic quiver $Q$.

4.4.4. $\text{Rep}_r(H_c, \mathbb{C}\Gamma)$. We have identified $\mu^{-1}(0)/\text{GL}(\delta)$ with $\text{Rep}_r(H_c, \mathbb{C}\Gamma)//\text{GL}(\mathbb{C}\Gamma)^\Gamma$ using Theorem 4.1 with $c = 0$. Let us see what happens for arbitrary $c$.

First of all, we claim that $H_c$ has no representations in $\mathbb{C}\Gamma$ if $c_1 \neq 0$. Indeed, $xy - yx$ acts in the same way as $c$. In particular, $c \in (\mathbb{C}\Gamma)^\Gamma$ has trace 0 on any $H_c$-module. But the trace of $c$ on $\mathbb{C}\Gamma$ is $|\Gamma|c_1$. From now on, we consider the case $c_1 = 0$.

Thanks to Theorem 4.1, $\mu^{-1}(\lambda)/\text{GL}(\delta)$ is identified with $\text{Rep}_r(H_c, \mathbb{C}\Gamma)//\text{GL}(\mathbb{C}\Gamma)^\Gamma$. Similarly to a remark above, we have a morphism $\text{Rep}_r(H_c, \mathbb{C}\Gamma)//\mathbb{C}\Gamma \to \text{Rep}(eH_c, e, \mathbb{C})$. We will see below that if $c_1 = 0$, then the algebra $eH_c$ is commutative. So $\text{Rep}(eH_c, e, \mathbb{C}) = \text{Spec}(eH_c)$. Also one can show that the morphism $\text{Rep}_r(H_c, \mathbb{C}\Gamma)//\mathbb{C}\Gamma \to \text{Rep}(eH_c, e, \mathbb{C})$ is
an isomorphism. We are not going to do this, but we will prove that \( \mu^{-1}(\lambda)/\GL(\delta) \cong \Spec(eH_c e) \) (perhaps for a different \( c' \)). Also we will see that the map \( z \mapsto e z \) from the center \( Z(H_c) \) of \( H_c \) to \( e H_c e \) is an isomorphism of algebras.

4.4.5. Deformations of \( \mathbb{C}^2/\Gamma \). Finally, let us remark that the algebras \( \mathbb{C}[\mu^{-1}(\lambda)/G] \) with \( \sum_{i=0}^r \delta_i \lambda_i = 0 \) (this is equivalent to \( c_1 = 0 \)) form an \( r \)-parametric deformation of \( \mathbb{C}^2/\Gamma \). Of course, all deformations obtained in this way are commutative. We will see below that one can produce non-commutative using a related construction called a quantum Hamiltonian reduction.

5. SYMPLECTIC QUOTIENT SINGULARITIES

5.1. Quotient singularities. We are interested in studying deformations of the algebras of the form \( S(V)^\Gamma \), where \( V \) is a vector space, \( S(V) \) is its symmetric algebra, and \( \Gamma \) is a finite subgroup of \( \GL(V) \). This algebra is the algebra of polynomial functions on the quotient \( V^*/\Gamma \). In fact, to get some non-trivial theory we will need to restrict the class of groups we are dealing with. Before we considered \( \Gamma \subset \SL_2(\mathbb{C}) \) so our first guess would be that we need \( \Gamma \subset \SL(V) \). However, this class is still too large. We will consider the case when \( V \) is a symplectic vector space, i.e., possesses a non-degenerate skew-symmetric form, say \( \omega \), and \( \Gamma \) preserves this form, i.e., lies in the symplectic group \( \Sp(V) \).

5.2. Poisson brackets. As usual, one of the reasons why we make this restriction is that this situation is easier. But there is a reason for that too. Roughly speaking, a (non-commutative) deformation of a commutative algebra gives rise to a new structure on this algebra, a Poisson bracket, and, for \( \Gamma \subset \Sp(V) \), the algebra \( S(V)^\Gamma \) already comes equipped with such a bracket.

Let us start with a general definition of a (Poisson bracket). Let \( A \) be a commutative associative unital algebra. A bracket on \( A \) is a skew-symmetric \( \mathbb{C} \)-bilinear map \( \{\cdot,\cdot\} : A \times A \to A \) satisfying the following two axioms, known as the Leibniz identity:

(L) \( \{a, bc\} = \{a, b\} c + \{a, c\} b \),
for all \( a, b, c \in A \). We remark, that thanks to \( \{\cdot,\cdot\} \) being skew-symmetric, we have also \( \{ab, c\} = a\{b, c\} + b\{a, c\} \). A bracket is called Poisson if it satisfies the Jacobi identity

(J) \( \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \),

An algebra equipped with a Poisson bracket is called a Poisson algebra.

Exercise 5.1. Let \( A \) be a commutative associative unital algebra.

(1) Let \( A \) be equipped with a bracket \( \{\cdot,\cdot\} \). Show that \( \{1, a\} = 0 \) for all \( a \in A \).
(2) Show that if \( a_1, \ldots, a_k \) are generators of \( A \), then there is at most one bracket \( \{\cdot,\cdot\} \) with given \( \{a_i, a_j\} \). Show that this bracket satisfies the Jacobi identity for all \( a, b, c \), if it does so for all \( a_i, a_j, a_k \).
(3) Finally, prove that if \( A = \mathbb{C}[a_1, \ldots, a_k] \), then a bracket exists for any values of \( \{a_i, a_j\} \) as long as \( \{a_i, a_j\} = -\{a_j, a_i\} \).

Let us proceed to examples. Let \( V, \omega \) and \( \Gamma \) be as above. Define \( \{\cdot,\cdot\} \) on \( S(V) \) by setting \( \{u, v\} := \omega(u, v) \) for \( u, v \in V \) and extending this bracket to the whole \( S(V) \) in a unique possible way. By the previous exercise we get a Poisson bracket, since \( \{\{u, v\}, w\} = 0 \) for all \( u, v, w \in V \).
Exercise 5.2. We can choose a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $V$ so that $\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \omega(y_i, x_j) = \delta_{ij}$. Let us identify $S(V)$ with $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Then $\{\cdot, \cdot\}$ is given by the formula

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i}.$$ 

Now, since $\Gamma \subset \text{Sp}(V)$ we have $\omega(\gamma u, \gamma v) = \omega(u, v)$ and hence $\{\gamma u, \gamma v\} = \{u, v\}$. We deduce that $\gamma$ leaves the bracket on $S(V)$ invariant, i.e., $\{\gamma f, \gamma g\} = \gamma \{f, g\}$ for all $\gamma \in \Gamma$, $f, g \in S(V)$. In particular, the subalgebra of invariants $S(V)^\Gamma$ is closed under $\{\cdot, \cdot\}$ and so it becomes a Poisson algebra.

Let us make a remark regarding a compatibility between the brackets and gradings. Assume that $A$ is graded, $A = \bigoplus_{n=0}^{\infty} A^n$. We say that $\{\cdot, \cdot\}$ has degree $-d$ if $\{A^i, A^j\} \subset A^{i+j-d}$. For example, the Poisson bracket on $S(V)$ (and hence also on $S(V)^\Gamma$ has degree $-2$).

Finally, let us discuss a connection between Poisson brackets and (filtered) deformations. Let $A$ be a filtered (associative unital) algebra. Assume that $A := \text{gr} A$ is commutative. This means that $[A^{\leq i}, A^{\leq j}] \subset A^{i+j-1}$. Let us pick a positive integer $d$ such that $[A^{\leq i}, A^{\leq j}] \subset A^{i+j-d}$. We can define a bracket of degree $-d$ on $A$ in a way similar to the definition of the product. Namely, pick $a \in A^i, b \in A^j$ and lift them to elements $\tilde{a} \in A^{\leq i}, \tilde{b} \in A^{\leq j}$. Then set $\{a, b\} := [\tilde{a}, \tilde{b}] + A^{\leq i+j-d-1}$, this is an element of $A^{i+j-d}$.

Exercise 5.3. Check that $\{\cdot, \cdot\}$ on $A = \text{gr} A$ is well-defined and is indeed a Poisson bracket.