LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS
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Correction to Section 15.4

Unfortunately, the argument in Section 15.4 of the lecture that shows that all \( n + 1 \) components of \( \mu^{-1}(0) \) contain a point with a trivial stabilizer is incorrect. The reason is that only \( \mathcal{M}_0, \mathcal{M}_n \) are subvarieties in \( \mu^{-1}(0) \), while the other \( n - 1 \) varieties \( \mathcal{M}_i \), even do not intersect \( \mu^{-1}(0) \). A correct argument is below.

Lemma 0.1. A generic representation in \( \text{Rep}(Q^{MK}, \delta) \) is indecomposable and its stabilizer in \( \text{GL}(\delta) \) is \( \mathbb{C}^\times \).

Proof. All subsets \( I(\alpha^1, \ldots, \alpha^n) \subset \text{Rep}(Q^{MK}, \delta) \) with \( n > 1 \) contain finitely many orbits. Let us decompose \( I(\delta) \) into locally closed irreducible \( G \)-stable subvarieties, \( I(\delta) = I^0(\delta) \cup I^1(\delta) \cup \ldots \cup I^k(\delta) \) with \( m(I^0(\delta)) = 1 \) and \( m(I^i(\delta)) = 0 \) for \( \delta > 0 \). This means that we may assume that all \( I^i(\delta) \) are single \( G \)-orbits. The dimension of every orbit does not exceed \( \dim \text{GL}(\delta) - 1 \).

We have \( \dim R = \dim \text{GL}(\delta) \). So we see that \( I^0(\delta) \) is dense in \( R \). Also a dimension count shows that a generic orbit in \( I^0(\delta) \) has to have dimension \( \dim \text{GL}(\delta) - 1 \). The stabilizer of every representation is connected (it is an open subset in the space of all endomorphisms of the representation). So we see that the stabilizer of a generic representation in \( I^0(\delta) \) is forced to coincide with \( \mathbb{C}^\times \), the kernel of the \( \text{GL}(\delta) \)-action.

Recall from Section 15.3, that the components of \( \mu^{-1}(0) \) are the closures of the conormal bundles to some locally closed subsets \( I^k \subset I(k\delta + \epsilon\infty, \delta, \ldots, \delta) \subset \text{Rep}(Q^{CM}, n\delta + \epsilon\infty) \) with \( m(I^k) = n \). We will now present such subsets. Namely, let \( z_1, \ldots, z_n \) be pairwise distinct indecomposable elements from \( \text{Rep}(Q^{MK}, \delta) \) with stabilizer \( \mathbb{C}^\times \). Such representations exist thanks to the previous lemma. Choose a decomposition \( \bigoplus_{i \in Q^{MK}} \mathbb{C}^n_{\delta_i} = \bigoplus_{i \in Q^{MK}} (\mathbb{C}^\times)^{\oplus n} \).

Then we can view \( x := \bigoplus_{i=1}^n x_i \) as an element of \( \text{Rep}(Q^{MK}, n\delta) \). Also we have the induced decomposition of the space \( \mathbb{C}^n \) sitting at the vertex 0 into the direct sum of one-dimensional subspaces. Let \( e_1, \ldots, e_n \) be a basis compatible with this decomposition. To get an element of \( \text{Rep}(Q^{CM}, n\delta + \epsilon\infty) \) from an element of \( \text{Rep}(Q^{MK}, n\delta) \) we need to add an element of \( \mathbb{C}^n \).

Set \( I^k := \{ x, i := \sum_{i=1}^k i \epsilon_i | i_1 \ldots i_k \neq 0 \} \), where \( x \) is as above. The stabilizer of \( x \) in \( G \) is \( \mathbb{C}^{\times n} \hookrightarrow \text{GL}(\delta)^{\times n} \hookrightarrow G = \text{GL}(n\delta) \). So the stabilizer of \( (x, i) \in I^k \) is \( \mathbb{C}^{\times(n-k)} \), the last \( n-k \) copies of \( \mathbb{C}^\times \) in \( \mathbb{C}^{\times n} \). We need to show that the stabilizer in \( \mathbb{C}^{\times(n-k)} \) of a generic point of the fiber in \( (x, i) \) of the conormal bundle to \( G(x, i) \) is trivial.

The space \( \mathfrak{g}(x, i) := T_x G(x, i) \) admits an epimorphism onto \( \mathfrak{g}x \) with kernel \( \mathfrak{g}_x i \). Clearly, \( \mathfrak{g}_x i = \text{Span}(e_1, \ldots, e_k) \). So the conormal space to the orbit \( G(x, i) \) naturally surjects onto \( (\mathbb{C}^n / \text{Span}(e_1, \ldots, e_k))^* \). The action of \( (\mathbb{C}^\times)^{n-k} \) on the latter space is faithful and so it is faithful on the whole conormal space implying, in particular, that the stabilizer of a generic point is trivial.