LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

13. Quantum CM systems and Rational Cherednik algebras

13.1. **Setting.** We pick an irreducible real reflection group W. Let $\mathfrak{h}$ be the complexification of the reflection representation of W, so $\mathfrak{h}$ is an irreducible W-module that comes equipped with a W-invariant symmetric form $(\cdot,\cdot)$. We set $n = \dim \mathfrak{h}$. Let S be the set of reflections in W. To each reflection $s \in W$ we assign a nonzero vector $\alpha_s \in \mathfrak{h}^*$, with $s\alpha_s = -\alpha_s$. Further we pick a function $c_\bullet : S \to \mathbb{C}$ that is constant on conjugacy classes. To the function c we can associate two objects, a quantum generalized Calogero-Moser system and a Rational Cherednik algebra.

First, let us recall the algebra of quantum observables for the system. Consider the algebra $D_\hbar(\mathfrak{h})$ that is the quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*)[\hbar]$ by the relations

$$[x,x']=0, [a,a']=0, [a,x]=\hbar\langle a,x\rangle, \quad x,x'\in\mathfrak{h}^*, a,a'\in\mathfrak{h}.$$

In this algebra we have the Vandermond element $\delta = \prod_{s\in S}\alpha_s$ and we can invert it getting the algebra $D_\hbar(\mathfrak{h}^{Reg}) = D_\hbar(\mathfrak{h})[\delta^{-1}]$. We remark that we have $[a,\delta^{-1}] = -\hbar\frac{\partial_a\delta}{\delta^2}$ and so the localization does make sense. Then on $D_\hbar(\mathfrak{h})$ we have an action of W by automorphisms that fixes $\hbar$ and on $\mathfrak{h},\mathfrak{h}^*$ is given as before. The element δ is W-sign-invariant and so the W-action extends to $D_\hbar(\mathfrak{h}^{Reg})$. The algebra of quantum observables will be $\mathcal{A}_\hbar := D_\hbar(\mathfrak{h}^{Reg})^W$.

We set $\Delta = \sum_{i=1}^n a_i^2 \in D_\hbar(\mathfrak{h})$, where the vectors $a_1,\ldots,a_n$ form an orthonormal basis. The Hamiltonian of interest, the Olshanetsky-Perelomov Hamiltonian, is given by

$$H = \Delta - \sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2} \in D_\hbar(\mathfrak{h}^{Reg}).$$

Further, we remark that Δ is W-invariant because so is $(\cdot,\cdot)$. Also

$$w.\frac{(\alpha_s,\alpha_s)}{\alpha_s^2} = \frac{(\alpha_{wsw^{-1}},\alpha_{wsw^{-1}})}{\alpha_{wsw^{-1}}^2}.$$

So the sum

$$\sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}$$

is also W-invariant and $H\in\mathcal{A}_\hbar$.

Our goal is to find pair-wise commuting elements $H_2,\ldots,H_{n+1}\in\mathcal{A}_\hbar$ that are algebraically independent modulo $\hbar$ such that $H = H_2$. The construction is based on Rational Cherednik algebras.

Recall that the Rational Cherednik algebra associated to $(\mathfrak{h},W)$ is the SRA for $\mathfrak{h}\oplus\mathfrak{h}^*, W$. We will need parameters c_s as before and the parameter t will be the independent variable $\hbar$, so we get an algebra over $\mathbb{C}[\hbar]$. The corresponding Cherednik algebra $H_{\hbar,c}$ is the quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*)[\hbar]\#W$ modulo the following relations, see Problem 6.10,

$$[x,x'] = 0 = [y,y'], [y,x] = \hbar\langle y,x\rangle - \sum_{s\in S}c_s\langle\alpha_s,y\rangle\langle\alpha_s^\vee,x\rangle s, \quad x,x'\in\mathfrak{h}^*, y,y'\in\mathfrak{h},$$

where $\alpha_s^\vee$ is a vector in $\mathfrak{h}$ with $s\alpha_s^\vee = -\alpha_s^\vee$ and $\langle \alpha_s^\vee, \alpha_s \rangle = 2$.

The main idea of constructing the elements $H_2, \ldots, H_{n+1}$ is as follows. Suppose that all c_s are 0. Then $H = \Delta$. Of course, in $D_h(\mathfrak{h})$ all elements from $S(\mathfrak{h}) \subset D_\hbar(W)$ commute with H. Only the W-invariant part $S(\mathfrak{h})^W$ is still present in $\mathcal{A}_\hbar$. We want to use the same strategy for the general c_s's. One shouldn't expect an embedding $S(\mathfrak{h}) \to D_\hbar(\mathfrak{h}^{Reg})$ such that $S(\mathfrak{h})^W$ commutes with H. Instead we will see that there is an embedding $S(\mathfrak{h}) \hookrightarrow D_\hbar(\mathfrak{h}^{Reg})\#W$ that works. Also there is a natural embedding $\mathbb{C}[\mathfrak{h}]\#W \hookrightarrow D_\hbar(\mathfrak{h}^{Reg})\#W$. The two embeddings will combine to a monomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$.

The required homomorphism will be constructed in two steps. We first construct a homomorphism $\Theta : H_{\hbar,c} \to D(\mathfrak{h}^{Reg})\#W$ that is essentially due to Dunkl. Then we will produce an an automorphism φ of $D_\hbar(\mathfrak{h}^{Reg})\#W$ such that $\varphi \circ \Theta$ has required properties.

13.2. **Dunkl operators.** For $a \in \mathfrak{h}$ we define an element $D_a \in D_\hbar(\mathfrak{h}^{Reg})\#W$ by

$$D_a = a - \sum_{s \in S} \frac{c_s \langle \alpha_s, a \rangle}{\alpha_s}(1 - s). \tag{1}$$

Clearly, D_a is linear in a.

Exercise 13.1. *Prove that $wD_aw^{-1} = D_{wa}$ for all $w \in W, a \in \mathfrak{h}$.*

We will see that a map $x \mapsto x, w \mapsto w, a \mapsto D_a$ extends to a homomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$. An important tool for this is a natural action of $D_\hbar(\mathfrak{h}^{Reg})\#W$ on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$.

The algebra $D_\hbar(\mathfrak{h}^{Reg})\#W$ acts on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$: the action of W is induced from the W-action on $\mathfrak{h}^{Reg}$, $f \in \mathbb{C}[\mathfrak{h}^{Reg}] \subset D_\hbar(\mathfrak{h}^{Reg})$ acts by the multiplication by f, and, finally, $a \in \mathfrak{h} \subset D_\hbar(\mathfrak{h}^{Reg})$ acts by $\hbar\partial_a$.

Lemma 13.1. *The representation of $D_\hbar(\mathfrak{h}^{Reg})\#W$ in $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ is faithful.*

Proof. The algebra $D_\hbar(\mathfrak{h}^{Reg})$ is a free module over $\mathbb{C}[\hbar]$, it is freely spanned by the vector space $S(\mathfrak{h}) \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^{Reg}]$. Also this algebra is graded "by the order of a differential operator": with $\mathbb{C}W$ and $\mathbb{C}[\mathfrak{h}^{Reg}]$ in degree 0, and $\hbar, \mathfrak{h}$ in degree 1. The action on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ is compatible with the grading and so the kernel is a graded ideal. A homogeneous element in $D_\hbar(\mathfrak{h}^{Reg})$ is 0 if its specialization at $\hbar = 1$ is 0. So it is enough to check that the representation of $D(\mathfrak{h}^{Reg})\#W$ in $\mathbb{C}[\mathfrak{h}^{Reg}]$ is faithful.

Let $d = \sum_{w \in W} d_w w$ with $d_w \in D(\mathfrak{h}^{Reg})$ lie in the kernel. Pick a point $x \in \mathfrak{h}^{Reg}$ and consider the completion $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx}$ of $\mathbb{C}[\mathfrak{h}^{Reg}]$ (or of $\mathbb{C}[\mathfrak{h}]$) at the ideal of Wx. The actions of both $D(\mathfrak{h}^{Reg})$ and W extend from $\mathbb{C}[\mathfrak{h}^{Reg}]$ to $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx}$. The action is continuous with respect to the inverse image topology on the completion. So d still acts by 0 on $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx}$. On the other hand, $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx} = \bigoplus_{w \in W} \mathbb{C}[\mathfrak{h}]^\wedge_{wx}$, where the latter is the completion at wx, a formal power series algebra. Each d_w preserves the summands, while W permutes them transitively. So each separate d_w acts by 0 on $\mathbb{C}[\mathfrak{h}]^\wedge_x$. We can represent d_w in the form $\sum_\alpha f_\alpha \partial^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha = a_1^{\alpha_1} \ldots a_n^{\alpha_n}$. Let $x^1, \ldots, x^n$ be the dual basis to $a_1, \ldots, a_n$. We can prove that $f_\alpha = 0$ by induction on $|\alpha| = \sum_i \alpha_i$ starting from $|\alpha| = 0$. For this we consider the action of d_w on the monomials in $x^1, \ldots, x^n$ of degree $|\alpha|$. The details are left to a reader. $\square$

We remark that, although D_a does not lie in $D_\hbar(\mathfrak{h})\#W$, it does preserve the subspace $\mathbb{C}[\mathfrak{h}][\hbar] \subset \mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$. This is because $f - s(f)$ is divisible by α_s for any $f \in \mathbb{C}[\mathfrak{h}]$.

Proposition 13.2. *The map $x \mapsto x, w \mapsto w, y \mapsto D_y, x \in \mathfrak{h}^*, w \in W, y \in \mathfrak{h}$ defines a $\mathbb{C}[\hbar]$-algebra homomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$.*

Proof. We need to show that the defining relations in the Cherednik algebra also hold in $D_\hbar(\mathfrak{h}^{Reg})\#W$. Obviously, two elements from $\mathfrak{h}^*$ in $D_\hbar(\mathfrak{h}^{Reg})\#W$ commute. Thanks to the previous exercise, the commutation relations between $\mathfrak{h}$ and W hold (for $\mathfrak{h}^*$ and W this is obvious). It remains to prove that

$$[D_y, x] = \langle y, x\rangle\hbar - \sum_{s\in S} c_s\langle y, \alpha_s\rangle\langle x, \alpha_s^\vee\rangle s. \tag{2}$$

$$[D_y, D_{y'}] = 0. \tag{3}$$

(2) reduces to $[\alpha_s^{-1}(1-s), x] = \langle x, \alpha_s\rangle s$. We can write x in the form $t\alpha_s + x_0$, where $sx_0 = x_0$. Then in the left hand side we get $\frac{1}{\alpha_s}(-s\alpha_s + \alpha_s s)t = 2st$ and in the right hand side we get the same.

The proof of (3) is more tricky. Thanks to Lemma 13.1, it is enough to check that $[D_y, D_{y'}]$ acts by 0 on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$. First of all, we claim that $[[D_y, D_{y'}], x] = 0$. To check this we use the Jacobi identity: $[[D_y, D_{y'}], x] = [D_y, [D_{y'}, x]] - [D_{y'}, [D_y, x]]$. By (2), we have

$$[D_y, [D_{y'}, x]] = [D_y, \hbar\langle y', x\rangle - \sum_{s\in S} c_s\langle y', \alpha_s\rangle\langle x, \alpha_s^\vee\rangle s] = \sum_{s\in S} c_s\langle y', \alpha_s\rangle\langle x, \alpha_s^\vee\rangle[s, D_y]. \tag{4}$$

But $[s, D_y] = sD_y - D_y s = (sD_y s^{-1} - D_y)s = (D_{sy} - D_y)s = -\langle\alpha_s, y\rangle D_{\alpha_s^\vee}s$. So we see that

$$[D_y, [D_{y'}, x]] = -\sum_{s\in S} c_s\langle y', \alpha_s\rangle\langle x, \alpha_s^\vee\rangle\langle y, \alpha_s\rangle D_{\alpha_s^\vee}.$$

This expression is symmetric in y and y' and so $[[D_y, D_{y'}], x] = [D_y, [D_{y'}, x]] - [D_{y'}, [D_y, x]] = 0$.

Also we remark that $D_y 1 = 0$ (the equality in $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$) and hence $[D_y, D_{y'}]1 = 0$. But, since $[[D_y, D_{y'}], x] = 0$, we see that, on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$, the bracket $[D_y, D_{y'}]$ commutes with $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ acting by multiplications. So $[D_y, D_{y'}]$ is zero on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ and therefore is zero as an element of $D_\hbar(\mathfrak{h}^{Reg})\#W$. □

Since D_y preserves $\mathbb{C}[\mathfrak{h}][\hbar]$ we get a representation of $H_{\hbar,c}$ on $\mathbb{C}[\mathfrak{h}][\hbar]$ (called the polynomial representation). This and more general representations, of the form $\mathbb{C}[\mathfrak{h}][\hbar]\otimes E$, where E is an irreducible W-module, play an important role in the representation theory of $H_{\hbar,c}$, these are analogs of Verma modules.

We remark that everything explained in this section works not only for real but also for complex reflection groups. One only needs to modify the definition of the Dunkl operator as follows. By α_s we denote an element in $\mathfrak{h}^*$ with non-unit eigenvalue, say λ_s. Then we define D_a by

$$D_a = a - \sum_{s\in S}\frac{2c_s\langle\alpha_s, a\rangle}{(1-\lambda_s)\alpha_s}(1-s).$$

Exercise 13.2. *Prove an analog of Proposition 13.2 for complex reflection groups.*

Exercise 13.3. *Let W be a complex reflection group.*

1. *Show that* $\operatorname{ad} f$ *is a locally nilpotent operator on $H_{\hbar,c}$ for any $f\in\mathbb{C}[\mathfrak{h}]^W$.*
2. *Deduce that the localization $H_{\hbar,c}[\delta^{-1}]$ exists.*
3. *Show that the Dunkl homomorphism $H_{\hbar,c}\to D_\hbar(\mathfrak{h}^{Reg})\#W$ factors through a unique homomorphism $H_{\hbar,c}[\delta^{-1}]\to D_\hbar(\mathfrak{h}^{Reg})\#W$.*
4. *Show that the homomorphism $H_{\hbar,c}[\delta^{-1}]\to D_\hbar(\mathfrak{h}^{Reg})\#W$ is an isomorphism.*

13.3. **OP Hamiltonian via Dunkl operators.** Let Θ denote the Dunkl homomorphism $H_{\hbar,c}\to D_\hbar(\mathfrak{h}^{Reg})\#W$.

Exercise 13.4. *Prove that Θ is injective modulo $\hbar$ and hence is injective.*

We denote the induced homomorphism $eH_{\hbar,c}e\to eD_\hbar(\mathfrak{h}^{Reg})\#We=D_\hbar(\mathfrak{h}^{Reg})^W$ also by Θ. We remark that there is a natural algebra homomorphism $S(\mathfrak{h})\to H_{\hbar,c}$ that is injective even modulo $\hbar$. This is because $H_{\hbar,c}$ is a free $\mathbb{C}[\hbar]$-module with spanned by $S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W=S(\mathfrak{h}^*)\otimes\mathbb{C}W\otimes S(\mathfrak{h})$. So we have a homomorphism $S(\mathfrak{h})^W\to eH_{\hbar,c}e\to D_\hbar(\mathfrak{h}^{Reg})^W$ that is injective modulo $\hbar$.

Lemma 13.3. *We have $\Theta(\Delta)=\overline{H}:=\Delta-\sum_{s\in S}c_s\frac{(\alpha_s,\alpha_s)}{\alpha_s}\alpha_s^\vee$.*

Proof. It follows from Lemma 13.1 that the algebra $D_\hbar(\mathfrak{h}^{Reg})^W=eD_\hbar(\mathfrak{h}^{Reg})\#We$ acts on $\mathbb{C}[\mathfrak{h}^{Reg}]^W[\hbar]=e\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$. So it is enough to compute $\sum_{i=1}^n D_{a_i}^2 f$, where $f\in\mathbb{C}[\mathfrak{h}^{Reg}]^W$. But $D_{a_i}f=a_if$ because $s(f)=f$ for all $s\in S$. Now

$$D_{a_i}a_if=a_i^2f-\sum_{s\in S}c_s\frac{\langle\alpha_s,a\rangle}{\alpha_s}(1-s)a_if=a_i^2f-\sum_{s\in S}c_s\frac{\langle\alpha_s,a_i\rangle}{\alpha_s}(a_i-s(a_i))sf=$$
$$(a_i^2-\sum_{s\in S}c_s\frac{\langle\alpha_s,a_i\rangle^2}{\alpha_s}\alpha_s^\vee)f.$$

To get the statement of the lemma, we need to sum the previous equalities for $i=1,\ldots,n$ and notice that $\sum_{i=1}^n\langle\alpha_s,a_i\rangle^2=(\alpha_s,\alpha_s)$ because $a_1,\ldots,a_n$ is an orthonormal basis. □

We will define a $\mathbb{C}[\hbar]$-algebra automorphism φ of $D_\hbar(\mathfrak{h}^{Reg})\#W$.

Exercise 13.5. *Define φ to be the identity on $\mathbb{C}[\mathfrak{h}^{Reg}]\#W$ and $\varphi(a)=a+\sum_{s\in S}c_s\frac{\langle a,\alpha_s\rangle}{\alpha_s}$. Show that φ extends to a $\mathbb{C}[\hbar]$-linear automorphism of $D_\hbar(\mathfrak{h}^{Reg})\#W$.*

Lemma 13.4. *We have $\varphi(\overline{H})=H$.*

Proof. We will prove that $\overline{H}=\varphi^{-1}(H)$, this is easier because

$$\varphi^{-1}(H)=\varphi^{-1}(\Delta)-\sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}.$$

We have

$$\varphi^{-1}(a_i^2)=(a_i-\sum_{s\in S}c_s\frac{\langle\alpha_s,a_i\rangle}{\alpha_s})^2=a_i^2-\sum_{s\in S}c_s\langle\alpha_s,a_i\rangle(a_i\alpha_s^{-1}+\alpha_s^{-1}a_i)+\sum_{s,s'\in S}c_sc_{s'}\frac{\langle\alpha_s,a_i\rangle\langle\alpha_{s'},a_i\rangle}{\alpha_s\alpha_{s'}}$$
$$=a_i^2-2\sum_{s\in S}c_s\langle\alpha_s,a_i\rangle\alpha_s^{-1}a_i+\hbar\sum_{s\in S}c_s\frac{\langle\alpha_s,a_i\rangle^2}{\alpha_s^2}+\sum_{s,s'\in S}c_sc_{s'}\frac{\langle\alpha_s,a_i\rangle\langle\alpha_{s'},a_i\rangle}{\alpha_s\alpha_{s'}}.$$

Summing over all i, we get

$$\varphi^{-1}(\Delta)=\Delta-\sum_{s\in S}\frac{c_s}{\alpha_s}\sum_{i=1}2\langle\alpha_s,a_i\rangle a_i+\hbar\sum_{s\in S}c_s\alpha_s^{-2}\sum_{i=1}^n\langle\alpha_s,a_i\rangle^2+$$
$$\sum_{s,s'}c_sc_{s'}\alpha_s^{-1}\alpha_{s'}^{-1}\sum_{i=1}^n\langle\alpha_s,a_i\rangle\langle\alpha_{s'},a_i\rangle=\Delta-\sum_{s\in S}\frac{c_s}{\alpha_s}(\alpha_s,\alpha_s)\alpha_s^\vee+$$
$$\sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}+\sum_{s\neq s'}c_sc_{s'}\frac{(\alpha_s,\alpha_{s'})}{\alpha_s\alpha_{s'}}.$$

To show that $\varphi^{-1}(H) = \overline{H}$ it remains to show that the last summand, say P, in the previous sum is 0. We remark that P is W-invariant and so δP is W-sign-invariant. Also δP is a polynomial of degree $\deg \delta - 2 = |S| - 2$. However, if $s(F) = -F$ for $F \in \mathbb{C}[\mathfrak{h}]$, then F is divisible by α_s. It follows that δP is divisible by δ, which is nonsense. $\square$

Summarizing, for the free generators $F_2 = \Delta, \ldots, F_{n+1}$ of $S(\mathfrak{h})^W$, the elements $H_i := \varphi(\Theta(F_i))$ form a completely integrable system (modulo $\hbar$, they generate a subalgebra isomorphic to $S(\mathfrak{h})^W$).

13.4. **Future directions.** We have found first integrals for the classical CM system of type A in two different ways: using Hamiltonian reduction and using the Dunkl homomorphism. One can ask how these two ways are related. The second way allowed us to deal with the quantum system as well. So another question is whether we can extend a Hamiltonian reduction procedure to deal with quantum systems.

These are basically questions that we will be dealing with in the next four lectures. We remark that we have already seen some kind of the answer to the first question in our study of Kleinian singularities. Namely, we have seen that the spherical SRA $eH_{0,c}e$ for the Kleinian group can be realized as a Hamiltonian reduction of the space $\mathrm{Rep}(Q, \delta)$ under the action of $GL(\delta)$ at a suitable point $\lambda \in \mathfrak{gl}(\delta)^{*\,\mathrm{GL}(\delta)}$, where Q is the double McKay quiver and δ is the imaginary root.

We will see that the algebra $eH_{t,c}e$ for $\Gamma = \mathfrak{S}_n \ltimes \Gamma_1^n$ can be realized as a *quantum* Hamiltonian reduction of the representation space of a suitable quiver.

Problem 13.6. *Let* $\Gamma = \mathfrak{S}_n$ *and* $\mathfrak{h} = \mathbb{C}^n$ *(and not the reflection representation, this is a minor technicality). The goal of this problem will be to relate the CM space* C *to* $\mathrm{Spec}(eH_{0,c}e)$*. We are going to produce a morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) /\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to C$*, to show that this is an isomorphism. Then we prove that the natural morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) /\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathrm{Spec}(eH_{0,c}e)$ *is an isomorphism.*

Let $y_1, \ldots, y_n$ *be the tautological basis in* $\mathbb{C}^n = \mathfrak{h}$ *and* $x_1, \ldots, x_n$ *be the dual basis in* $\mathfrak{h}^*$*. The elements* x_n, y_n *still act on* $N^{\mathfrak{S}_{n-1}} \cong \mathbb{C}^n$*. Show that* $[x_n, y_n] \in O = \{A | \operatorname{tr} A = 0, \operatorname{rk}(A+E) = 1\}$ *for a suitable choice of* c*. Deduce that we have a morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) \to \mu^{-1}(O)$*. Show that it descends to a morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) /\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to C$*. Show that the latter is finite and birational. Deduce that it is an isomorphism.*

Show that a natural morphism $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) /\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathrm{Spec}(eH_{0,c}e)$ *(how is it constructed, by the way?) is also finite and birational. Deduce that it is an isomorphism.*