PSet 1. Solutions

Pr 1:

1) Let $v \in V$ be a generator. Then $A \to V$, $a \mapsto av$ is an epimorphism. Since A is countable dimensional, then so is V.

2) An element $\varphi \in End_A(V)$ is completely determined by its behavior on v. So $\varphi \mapsto \varphi(v)$ is an inclusion $End_A(V) \hookrightarrow V$. So $End_A(V)$ is at most countable dimensional.

3) $\varphi - z$ is an endomorphism of V. So its kernel and image are submodules. Since V is irreducible, we deduce that $\varphi - z$ is injective and surjective, hence automorphism. Further assume that there are different $z_1, \ldots z_n \in \mathbb{F}$ and $a_1 \ldots a_n \in \mathbb{F}$ s.t. $\sum_{i=1}^{n} a_i (\varphi - z_i)^{-1} = 0$. So $\sum_{i=1}^{n} a_i \prod_{j \neq i} (\varphi - z_j) = 0$. The l.h.s is a polynomial $F(z)$. It decomposes into $\alpha \prod_{k=1}^{\ell} (\varphi - u_k)$ provided not all a_i are zero. Since all $\varphi - u_k$ are automorphisms we arrive at a contradiction. We conclude that the elements $(\varphi - z)^{-1}$, $z \in \mathbb{F}$, are linearly independent.

4) Assume $End_A(V) \neq \mathbb{F}$. Then there is a non-constant element $\varphi \in End_A(V)$. On Step 3 we have established an uncountable collection of linearly independent elements of $End_A(V)$. Contradiction w. Step 2.

Problem 2:

1) Clearly, $X_iX_jf = X_jX_if$. It is easy to see that $T_i^2 f = f$ and $T_iT_jf = T_jT_if$, $|i-j|>1$, $X_iT_jf = T_jX_if$, $j \neq i, i+1$

Let us check that $X_{i+1}T_if = T_iX_if + f$ (one needs to fix the formula for T_if: $T_if = s_if - \frac{s_if-f}{x_{i+1}-x_i}$)

$$T_iX_if = X_{i+1}s_if - \frac{X_{i+1}s_if - X_if}{x_{i+1}-x_i}$$

$$X_{i+1}T_if = X_{i+1}s_if - X_{i+1}\frac{s_if-f}{x_{i+1}-x_i} = X_{i+1}s_if - \frac{x_{i+1}s_if - x_{i+1}f}{x_{i+1}-x_i} = T_iX_if - \frac{x_{i+1}f - x_if}{x_{i+1}-x_i}$$

Checking $T_iT_{i+1}T_if = T_{i+1}T_iT_{i+1}f$ directly is complicated. Instead we will check this on monomials using induction of degree. Clearly, $T_iT_{i+1}T_i1 = T_{i+1}T_iT_{i+1}1$

Now suppose $T_iT_{i+1}T_if = T_{i+1}T_iT_{i+1}f$ and check $T_iT_{i+1}T_iX_jf = T_{i+1}T_iT_{i+1}X_jf$. This is easy if $j \neq i, i+1, i+2$, because here X_j commutes with T_i, T_{i+1}. Let's consider $j=i$, other 2 cases are similar: (here we use only rel-ns that have been already established)

$T_iT_{i+1}T_iX_if = T_iT_{i+1}(X_{i+1}T_i - 1)f = T_i(X_{i+2}T_{i+1} - 1)T_if - T_iT_{i+1}f$
$= X_{i+2}T_iT_{i+1}T_if - f - T_iT_{i+1}f$

$T_{i+1}T_iT_{i+1}X_if = T_{i+1}T_iX_iT_{i+1}f = T_{i+1}(X_{i+1}T_i - 1)T_{i+1}f = (X_{i+2}T_{i+1} - 1)T_iT_{i+1}f$
$- f = T_iT_{i+1}T_iX_if$

2) Suppose $\sum_{\sigma \in S_d} f_\sigma \otimes \sigma$ acts trivially on $\mathbb{C}[x_1, \dots x_d]$.

Considering the top degree part of $\sum_{\sigma \in S_d} f_\sigma \sigma(F) = 0$ we see that $\sum_{\sigma \in S_d} \bar{f}_\sigma \sigma F$ (where σF is obtained from F by permuting variables) is zero. Here $\bar{f}_\sigma$ is the top degree part of f_σ ~~We will apply this~~ (we take the largest degree appearing in all f_σ's). So we can assume that we can find homogeneous polynomials f_σ of the same degree such that $\sum_{\sigma \in S_n} f_\sigma \sigma(F) = 0$ for any monomial F. ~~We may~~

Order monomials in f_σ lexicographically. Replacing $\sum_{\sigma \in S_n} f_\sigma \otimes \sigma$ with $\sum_{\sigma \in S_n} f_\sigma \otimes \sigma\tau$, we may assume that the largest monomial occurs in f_1, let $x_1^{m_1} \cdots x_\alpha^{m_\alpha}$ be this monomial. Take $F = x_1^{\alpha-1} x_2^{\alpha-2} \cdots x_{\alpha-1}$. Then the monomial $x_1^{m_1+\alpha-1} x_2^{m_2+\alpha-2} \cdots x_\alpha^{m_\alpha}$ occurs in $f_1 F$, while all monomials in $f_\sigma F$ are strictly smaller. So $\sum_{\sigma \in S_n} f_\sigma \sigma(F) \neq 0$. Contradiction.

3) Since $\varphi \circ (i_1 \otimes i_2)$ is injective, $i_1 \otimes i_2$ is injective. On the other hand, thanks to relations between T_i's and X_j's, we can write any element of $\mathcal{H}(\alpha)$ with X's in front. So $i_1 \otimes i_2$ is surjective

Pr 3:

1) It is easy to check this relation for the images of T_i, F in $\mathbb{C}[x_1, .. x_d]$. Since the latter is faithful, we are done

2) We have $X_i F = F X_i$ and $s_i F = F$, hence $T_i F = F T_i$ by part 1

3) Let M be an irreducible $\mathcal{H}(d)$-module. The subalgebra $\mathbb{C}[X_1, .. X_d]^{S_d}$ is central and, by the Schur lemma, acts on M by a character, say χ. Let $\mathcal{H}(d)_\chi$ denote the corresponding central reduction. Note that $\mathbb{C}[X_1, .. X_d]$ is a free $\mathbb{C}[X_1, .. X_d]^{S_d}$-module of rank $d!$ It follows from Prob 2 that $\mathcal{H}(d)$ is a free $\mathbb{C}[X_1, .. X_d]^{S_d}$-module of rank $(d!)^2$. So $\dim \mathcal{H}(d)_\chi = (d!)^2$. Since it is finite, we have $\dim M < \infty$

4) Moreover $\mathcal{H}(d)_\chi \twoheadrightarrow End(M)$ (Burnside thm). So $\dim M \leq d!$

5) Let χ be the character of evaluation at a point $(a_1, .. a_d)$, where $a_i - a_j \notin \{0, \pm 1\}$. We are going to show that $\mathcal{H}(d)_\chi \simeq Mat_{d!}(\mathbb{C})$. First of all, since $a_i \neq a_j$, we see that $\mathbb{C}[X_1 .. X_d] / (\mathbb{C}[X_1, .. X_d]^{S_d}_\chi) \simeq \mathbb{C}^{\oplus d!}$. By Problem 2, $\mathcal{H}(d)_\chi$ acts on $\mathbb{C}[X_1, .. X_d]/(\mathbb{C}[X_1, .. X_d]^{S_d}_\chi)$ so that the latter acts by left multiplications. Note that since $a_i \neq a_{i+1} \pm 1$, $s_i(a_1, .. a_d)$ is also a character for any submodule $M' \subset \mathbb{C}[X_1, .. X_d]/(\mathbb{C}[X_1, .. X_d]^{S_d}_\chi)$ (for the action of $X_1, .. X_d$; here we assume that $(a_1, .. a_d)$ is a character of M'). This follows from the representation theory of $\mathcal{H}(2)$. From the choice of $(a_1, .. a_d)$, we conclude that $M' = \mathbb{C}[X_1 .. X_d]/(\mathbb{C}[X_1 .. X_d]^{S_d}_\chi)$

Now we can prove the original claim. Let $U \subset Spec(\mathbb{C}[X_1, .. X_d]^{S_d})$ be the open subset specified by $a_i \neq a_j, a_j \pm 1$. Consider the algebra $\mathcal{H}(d)_U = \mathbb{C}[U] \otimes_{\mathbb{C}[X_1, .. X_d]^{S_d}} \mathcal{H}(d)$. First, let's check that $Z(\mathcal{H}(d)_U) = \mathbb{C}[U]$. Otherwise, there is $u \in U$ such that the projection of $Z(\mathcal{H}(d)_U)$ to $\mathcal{H}(d)_u$ is of dimension more than 1. But this projection is central, that contradicts the previous paragraph. Now $Z(\mathcal{H}(d)) = \mathcal{H}(d) \cap Z(\mathcal{H}(d)_U)$ (intersection inside $\mathcal{H}(d)_U$). This equals $\mathcal{H}(d) \cap \mathbb{C}[U] = \mathbb{C}[X_1, .. X_d]^{S_d}$.

Problem 4

a) Let $\lambda \vdash n-1$, $\mu \vdash n$ be two partitions (Young diagrams)

If $V_\lambda \to V_\mu$, then we pick $v_p \in V_\lambda \subset V_\mu$. If $(w_1, \dots w_n)$ is its weight as an element of V_μ, then its weight in V_λ is $(w_1, \dots w_{n-1})$. The tableau on μ giving $(w_1 \dots w_n)$ is obtained from the tableau on λ giving $(w_1, \dots w_{n-1})$ by adding a box with content w_n. So $\lambda \to \mu$ in the Young graph

If $\lambda \to \mu$, then we can find a tableau T on μ with n in $\mu \setminus \lambda$. Let T' be a tableau obtained from T by removing that box. Let w', w be corresponding weights. The vector V_T with weight w is forced to lie in V_λ because $(w_1, \dots w_{n-1}) = w'$ is a weight of V_λ. So $V_\lambda \to V_\mu$

b) First, let's notice that $\lambda \to \lambda^t$ is an automorphism. Let's show that there's nothing else (and so the autom. group is $\mathbb{Z}/2\mathbb{Z}$)

Note that $\square$ is connected to two diagrams ($\square\square$, $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$) and any other diagram is connected to at least 3 other. So, if φ is an automorphism, then it fixes $\square$ and preserves $\square\square$, $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$. Replacing φ w. φ^t if needed, we may assume that φ maps $\square\square$ to $\square\square$, $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ to $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$. Now we prove that $\varphi(\lambda)$ using induction on $|\lambda|$ (total # of boxes). Clearly, $|\varphi(\lambda)| = |\lambda|$ (this the length of a path from $\square$ to λ $(+1)$). By induction, φ preserves all diagrams obtained from λ by removing a box. So these diagrams for λ & $\varphi(\lambda)$ are the same. If λ is not a rectangle, then it's a union of all diagrams obtained from λ by removing a box. Since $|\lambda| > 2$, we can also recover a rectangle from the only diagram obtained from it by removing a box. This completes induction step