REPRESENTATION THEORY, PROBLEM SET 5

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The deadline for submitting the solutions is Dec 14, 12pm. The solutions are to be submitted electronically (scanned hand-written solutions are fine). E-mail i.loseu@neu.edu.

There are three problems with total number of points equal to 30. The maximal number of points you get for this problem set is 20. 50% of your score above 20 will count to augment your scores from previous psets. Partial credit is given.

Problem 1. Prove the Krull-Schmidt theorem (Theorem 1.1 from Lecture 16) (5pts).

Problem 2. Suppose a quiver Q contains no oriented cycles. Then the dimension of an irreducible representation is a simple root and the number of isomorphism classes of irreducible representations coincides with the number of vertices (3pts).

Problem 3. Prove the Kac theorem using linear algebra for the following quivers:

1) Type A_m quiver oriented left to right (2pts).

2,3) Quivers a),d) from Pic. 1.2 to Lecture 16, 2 pts each.

4) The Kroneker quiver with two vertices and two arrows in the same direction. This case is harder.

Problem 4. Let G be a connected algebraic group acting linearly on a vector space V with finitely many orbits. Show that G acts on V^* with finitely many orbits and that the number of orbits in V^* coincides with the number of orbits in V (4pts).

Problem 5. Check if there is a solution to the DS problem in the following cases (2pts per case). In which cases the number of solutions $Y_1, \ldots, Y_k$ is finite (2pts)? Below we indicate the dimension of the space and representatives of conjugacy classes.

a) $n = 2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

b) $n = 4, \operatorname{diag}(1, 1, 0, 0), \operatorname{diag}(-1, -1, 0, 0), \operatorname{diag}(2, 2, 0, 0), \operatorname{diag}(-2, -2, 0, 0)$.

c) $n = 4, \operatorname{diag}(2a, 0, 0, 0), \operatorname{diag}(b, b, 0, 0), \operatorname{diag}(c, c, 0, 0), \operatorname{diag}(d, d, e, e)$, where a, b, c, d, e are generic with $a + b + c + d + e = 0$.

d) $n = 4, C_1 = O_{(2,1,1)}, C_2 = C_3 = C_4 = O_{(2,2)}$ (where the notation O_λ means the nilpotent orbit corresponding to the diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$).