

## REPRESENTATION THEORY, PROBLEM SET 4

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The deadline for submitting the solutions is Nov 23, 12pm. The solutions are to be submitted electronically (scanned hand-written solutions are fine). E-mail i.loseu@neu.edu.

There are three problems with total number of points equal to 30. The maximal number of points you get for this problem set is 20. Everything above 20 does not count. Partial credit is given.

**Problem 1.** This problem describes central elements in  $U_q(\mathfrak{sl}_2)$ .

- 1) Prove that the quantum Casimir element  $C = FE + \frac{Kq+K^{-1}q^{-1}}{(q-q^{-1})^2}$  is central (3pts).
- 2) Let  $q$  be a primitive  $d$ th root of 1. Set  $d_0 = d$  if  $d$  is odd and  $d_0 = d/2$  if  $d$  is even. Then the elements  $E^{d_0}, K^{d_0}, F^{d_0}$  are central (3pts).

**Problem 2.** This problem deals with the tautological representation of  $U = U_q(\mathfrak{sl}_n)$ , its R-matrix, and the link invariant known as HOMFLY polynomial. Let  $E_i, K_i^{\pm 1}, F_i, i = 1, \dots, n-1$ , denote the generators of  $U_q(\mathfrak{sl}_n)$ .

1) Show that  $E_i \mapsto E_{i,i+1}, F_i \mapsto E_{i,i+1}, K_i \mapsto \text{diag}(1, \dots, 1, q, q^{-1}, 1, \dots, 1)$  (where  $q$  appears in the  $i$ th place) defines a representation of  $U_q(\mathfrak{sl}_n)$  (2pts). This representation is called tautological and will be denoted by  $V$ .

2) Define an automorphism  $R$  of  $V \otimes V$  as follows (by  $v_1, \dots, v_n$  we denote the tautological basis of  $V$ ):

$$R(v_i \otimes v_j) = \begin{cases} q^{-1}v_i \otimes v_j, & \text{if } i = j, \\ v_i \otimes v_j, & \text{if } i < j, \\ v_i \otimes v_j + (q^{-1} - q)v_j \otimes v_i, & \text{if } i > j. \end{cases}$$

Show that it intertwines  $\Delta(u)$  and  $\Delta(u)^{op}$  (2pts).

3) Show that  $R$  satisfies QYBE. Moreover,  $\tau = R \circ \sigma$  satisfies the Hecke relation  $\tau^2 = (q^{-1} - q)\tau + 1$  (4pts; 2pts if only the case of  $\mathfrak{sl}_2$  is done). So we get the representation  $\Phi'_n$  of  $B_n$  in  $V^{\otimes n}$ .

4) Consider the endomorphism  $\mathcal{K}$  of  $V$  given by  $\mathcal{K}v_i = q^{n+1-2i}v_i$ . Prove that  $\varphi_m(b) = q^{n \deg(b)} \text{tr}(\Phi'_m(b)\mathcal{K}^{\otimes m})$ ,  $m \in \mathbb{Z}_{>0}$ , is a Markov trace (5pts; 3pts if only the case of  $\mathfrak{sl}_2$  is done).

5) Deduce from 4) that there is a link invariant  $P(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}, (q - q^{-1})^{-1}]$  that sends the unlink  $\mathbb{S}^1 \sqcup \dots \sqcup \mathbb{S}^1 \rightarrow \mathbb{R}^3$  ( $k$  circles) to  $\left(\frac{a-a^{-1}}{q-q^{-1}}\right)^k$  and satisfies  $a^{-1}P(L_+) - aP(L_-) = (q^{-1} - q)P(L_0)$ . This invariant is called the HOMFLY polynomial (3pts).

**Problem 3.** Prove Theorem 1.9 from Lecture 15 (the classification of the finite dimensional irreducible representations of  $\dot{U}_\epsilon(\mathfrak{sl}_2)$ ).

a) Show that the irreducible  $u_\epsilon$ -module  $L(\kappa\epsilon^r)$ , where  $r \in \{0, \dots, d-1\}$  extends to (an automatically irreducible)  $\dot{U}_\epsilon$ -module, where  $E^{(d)}, F^{(d)}$  act by 0 (2pts).

b) Show that  $L(\kappa, r) \otimes \text{Fr}^*L(m)$  has a required vector  $v_{\kappa, n}$  (2pts).

c) Show that  $L(\kappa, r) \otimes \text{Fr}^*L(m)$  is irreducible (2pts).

d) Show that any finite dimensional irreducible  $\dot{U}_\epsilon$ -module is isomorphic to a unique  $L(\kappa, r) \otimes \text{Fr}^* L(m)$  (2pts).