PSet 4 Solutions

(1) a) $KCK^{-1}=C$

$$[E,C]=[E,FE]+\left[E,\frac{Kq+K^{-1}q^{-1}}{(q-q^{-1})^2}\right]=[E,F]E+E\frac{Kq+K^{-1}q^{-1}}{(q-q^{-1})^2}-\frac{Kq+K^{-1}q^{-1}}{(q-q^{-1})^2}E$$

$$=\frac{K-K^{-1}}{q-q^{-1}}E+\frac{Kq^{-1}+K^{-1}q}{(q-q^{-1})^2}E-\frac{Kq+K^{-1}q^{-1}}{(q-q^{-1})^2}E=0$$

Similar computation shows $[F,C]=0$

b) $K^{\alpha_0}EK^{-\alpha_0}=q^{2\alpha_0}E=E$, $K^{\alpha_0}FK^{-\alpha_0}=q^{-2\alpha_0}F=F \Rightarrow K^{\alpha_0}$ is central

$KE^{\alpha_0}K^{-1}=q^{2\alpha_0}E^{\alpha_0}=E^{\alpha_0}$, $[F,E^{\alpha_0}]=[(1.2)$ in lec 14$]=-[\alpha_0]_q[K;1-\alpha_0]$

$E^{\alpha_0-1}=0 \Rightarrow E^{\alpha_0}$ is central. Similarly, we see that F^{α_0} is central.

(2) a) All matrices K_i are diagonal so (ii_q) holds. Note that $K_i \mapsto q^{h_i}$, where $h_i=(0,..0,1,-1,..0)$ so (i_q) follows from the corresponding relations for $\mathfrak{sl}_n$ and so does (iii_q) (note that $\frac{K_i-K_i^{-1}}{q-q^{-1}}$ goes to h_i) (iv_q) follows from the relations for $\mathfrak{sl}_n$. If $a_{ij}=0$ ($\Leftrightarrow |i-j|>1$), then $E_iE_j, F_iF_j, E_jE_i, E_jE_i$ go to 0. If $|i-j|=1$, then $E_i^2E_j, E_iE_jE_i, E_jE_i^2$, $F_i^2F_j, F_iF_jF_i, F_jF_i^2$ all go to 0 (direct computation), hence (v_q), (vi_q)

b) $R\Delta^{op}(K_i)=\Delta(K_i)R$ is straightforward. Let us check that $R\Delta^{op}(E_i)v_j\otimes v_k=\Delta(E_i)R(v_j\otimes v_k)$. If $i+1\notin\{j,k\}$, then both sides are 0. If $\{j,k\}\subset\{i,i+1\}$, then the equality follows from the case of $\mathfrak{sl}_2$ considered in Lecture 13. So we need to consider two cases: $j=i+1, k\notin\{i,i+1\}$ and $k=i+1, j\notin\{i,i+1\}$

We have $\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i$, $\Delta^{op}(E_i)=1\otimes E_i+E_i\otimes K_i$

$$R\Delta^{op}(E_i)v_{i+1}\otimes v_k=Rv_i\otimes v_k=v_i\otimes v_k+\delta_{i>k}(q^{-1}-q)v_k\otimes v_i$$

$$\Delta(E_i)R\,v_{i+1}\otimes v_k=(E_i\otimes 1+K_i\otimes E_i)(v_{i+1}\otimes v_k+\delta_{i+1>k}(q^{-1}-q)v_k\otimes v_{i+1})$$

$=v_i\otimes v_k+\delta_{i+1>k}(q^{-1}-q)v_k\otimes v_i$. Since $k\notin\{i,i+1\}$, we have $\delta_{i>k}=\delta_{i+1>k}$ and the required equality follows. The case $k=i+1, j\notin\{i,i+1\}$ as well as the equality $R\Delta^{op}(F_i) = \Delta(F_i)R$ are analogous $\square$

3) We can write $R(v_i\otimes v_j) = a_{ij}\, v_i\otimes v_j + b_{ij}\, v_j \otimes v_i$ with $a_{ij} = q^{-\delta_{ij}}$, $b_{ij} = \delta_{i>j}(q^{-1}-q)$

Let's check QYBE:

$R_{12}R_{13}R_{23}(v_i\otimes v_j\otimes v_k) = R_{12}R_{13}(a_{jk}\, v_i\otimes v_j\otimes v_k + b_{jk}\, v_i\otimes v_k\otimes v_j) = R_{12}(a_{ik}a_{jk}\, v_i\otimes v_j\otimes v_k$
$+ b_{ik}a_{jk}\, v_k\otimes v_j\otimes v_i + a_{ij}b_{jk}\, v_i\otimes v_k\otimes v_j + b_{ij}b_{jk}\, v_j\otimes v_k\otimes v_i) =$
(1) $a_{ij}a_{ik}a_{jk}\, v_i\otimes v_j\otimes v_k +$ (2) $b_{ij}a_{ik}a_{jk}\, v_j\otimes v_i\otimes v_k + a_{kj}^2 b_{ik}\, v_k\otimes v_j\otimes v_i + b_{kj}b_{ik}a_{jk}\, v_j\otimes v_k\otimes v_i$
$+$ (3) $a_{ik}a_{ij}b_{jk}\, v_i\otimes v_k\otimes v_j + b_{ik}a_{ij}b_{jk}\, v_k\otimes v_i\otimes v_j + a_{jk}b_{ij}b_{jk}\, v_j\otimes v_k\otimes v_i +$ (4) $b_{ij}b_{jk}^2\, v_k\otimes v_j\otimes v_i$

$R_{23}R_{13}R_{12}(v_i\otimes v_j\otimes v_k) = R_{23}R_{13}(a_{ij}\, v_i\otimes v_j\otimes v_k + b_{ij}\, v_j\otimes v_i\otimes v_k) = R_{23}(a_{ik}a_{ij}\, v_i\otimes v_j\otimes v_k +$
$+ b_{ik}a_{ij}\, v_k\otimes v_j\otimes v_i + a_{jk}b_{ij}\, v_j\otimes v_i\otimes v_k + b_{jk}b_{ij}\, v_k\otimes v_i\otimes v_j) =$
(1) $a_{ij}a_{ik}a_{jk}\, v_i\otimes v_j\otimes v_k +$ (3) $b_{jk}a_{ik}a_{ij}\, v_i\otimes v_k\otimes v_j + b_{ik}a_{ij}^2\, v_k\otimes v_j\otimes v_i + b_{ji}b_{ik}a_{ij}\, v_k\otimes v_i\otimes v_j$
$+$ (2) $a_{ik}a_{jk}b_{ij}\, v_j\otimes v_i\otimes v_k + b_{ik}a_{jk}b_{ij}\, v_j\otimes v_k\otimes v_i + a_{ki}b_{jk}b_{ij}\, v_k\otimes v_i\otimes v_j +$ (4) $b_{jk}b_{ij}^2\, v_k\otimes v_j\otimes v_i$

The summands marked (1)-(4) in 2 sums are the same (note that $b_{ij}b_{jk}^2 = b_{ij}^2 b_{jk}$)

So the difference $R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12}(v_i\otimes v_j\otimes v_k) = S_1 + S_2 + S_3$, where

$S_1 = (b_{kj}b_{ik} + b_{ij}b_{jk} - b_{ij}b_{ik})a_{jk}\cdot v_j\otimes v_k\otimes v_i$

$S_2 = (b_{ik}b_{jk} - b_{ik}b_{ji} - b_{jk}b_{ij})\,a_{ij}\cdot v_k\otimes v_i\otimes v_j$

$S_3 = b_{ik}(a_{jk}^2 - a_{ij}^2)\, v_k\otimes v_j\otimes v_i$

Let's simplify S_1. $b_{kj}b_{ik} + b_{ij}b_{jk} - b_{ij}b_{ik} = (q^{-1}-q)^2(\delta_{i>k>j} + \delta_{i>j>k} - \delta_{i>j,k})$
$= -\delta_{i>k}\delta_{kj}(q^{-1}-q)^2$. So $S_1 = -\delta_{i>k}\delta_{jk}\, q^{-1}(q^{-1}-q)^2\, v_j\otimes v_k\otimes v_i$

Similarly, $S_2 = \delta_{i>k}\delta_{ij}\, q^{-1}(q^{-1}-q)^2\, v_k\otimes v_i\otimes v_j$. Finally, we have

$$S_3 = \begin{cases} 0, & j\notin\{k,i\} \text{ or } i=j=k \text{ or } k\leq i \\ (q^{-1}-q)(q^{-2}-1)\, v_j\otimes v_k\otimes v_i & \text{if } i>j=k \\ (q^{-1}-q)(1-q^{-2})\, v_k\otimes v_i\otimes v_j & \text{if } i=j>k \end{cases}$$

We conclude that $S_1+S_2+S_3=0$ that finishes the proof of QYBE

Let's now show the Hecke relation: $(\tau-q^{-1})(\tau+q)=0$. We have

$$(\tau-q^{-1})(v_i\otimes v_j) = \begin{cases} v_j\otimes v_i - q^{-1}\, v_i\otimes v_j, & i<j \\ v_j\otimes v_i - q\, v_i\otimes v_j, & i>j \end{cases}, \text{ 0 else}$$

$$(\tau+q)(v_i\otimes v_j) = \begin{cases} (1+q)\, v_i\otimes v_j, & i=j \\ v_j\otimes v_i + q\, v_i\otimes v_j, & i<j \\ v_j\otimes v_i + q^{-1} v_i\otimes v_j, & i>j \end{cases}$$

The equality $(\tau-q^{-1})(\tau+q)\, v_i\otimes v_j = 0$ is now straightforward

4) Let's check (M1). Recall that $\phi'_m(b)$ commutes w. $K^{\otimes m}$ – that's how K acts on $V^{\otimes m}$. So ~~$\varphi_m(ab)$~~ $\varphi_m(ab) = q^{n\deg(ab)}\, tr(\phi'_m(a)\phi'_m(b)K^{\otimes m})$
$= q^{n\deg(ba)}\, tr(\phi'_m(a)K^{\otimes m}\phi'_m(b)) = q^{n\deg(ba)}\, tr(\phi'_m(b)\phi'_m(a)K^{\otimes m}) = \varphi_m(ba)$

Let's now check (M2): $\varphi_{m+1}(bT_m) = \varphi_m(b)$ (for T_m^{-1} the check is completely analogous). Set $\Phi_m(b) = q^{n\deg(b)}\phi'_m(b)K^{\otimes m}$ so that
$\Phi_{m+1}(b) = (\Phi_m(b)\otimes id_V)\circ(id_V^{\otimes m}\otimes K)\, \underbrace{q^n T_n}_{= id_V^{\otimes m-1}\otimes(R\circ\sigma)}$

For $u\in V^{\otimes m-1}$ we can write $\Phi_m(b)$ as $\Phi_m(b)(u\otimes v_i) = \sum_{j=1}^n A_j^i(u)\otimes v_j$. So $\varphi_m(b) = \sum_{i=1}^n tr(A_i^i)$. Now let us compute $\varphi_{m+1}(b)$. Apply $\Phi_{m+1}(bT_n)$ to $v_i\otimes v_j$:

$q^n T_m(u\otimes v_i\otimes v_j) = q^n\, \cancel{R(u\otimes}\; u\otimes K(v_j\otimes v_i) = q^n u\otimes(q^{-\delta_{ij}} v_j\otimes v_i + \delta_{j>i}(q^{-1}-q)\, v_i\otimes v_j)$

(*) $(id_V^{\otimes m}\otimes K)\, q^n T_m(u\otimes v_i\otimes v_j) = q^n q^{n+1-2i} q^{-\delta_{ij}}\, u\otimes v_j\otimes v_i + q^n q^{n+1-2j}\delta_{j>i}(q^{-1}-q)\, u\otimes v_i\otimes v_j$

In the computation of the trace we only care about components A_{ij}^{ij} defined by $\Phi_{m+1}(bT_n)u\otimes v_i\otimes v_j = \sum A_{kl}^{ij}(u)\otimes v_i\otimes v_j$. Such a component is zero when $i>j$ by (*). When $i=j$, by (*) we have

$A_{ii}^{ii}(u) = q^n q^{n+1-2i} q^{-1} A_i(u) = q^{2n-2i} A_i(u)$

When $j>i$, then we have $A_{ij}^{ij}(u) = q^{2n+1-2j}(q^{-1}-q)A_i(u)$

So $\varphi_{m+1}(bT_n) = \sum_i q^{2n-2i}\, tr A_i + \sum_i \sum_{j>i}(q^{2n-2j} - q^{2n+2-2j})\, tr(A_i) =$
$= \sum_i tr(A_i) = \varphi_m(b)$

So $\varphi_m(b)$ indeed form a Markov trace

5) We can present L_+, L_-, L_0 by braids of the form $b_1T_ib_2$, $b_1T_i^{-1}b_2$, b_1b_2. Recall the Hecke rel-n for the action of T_i: $T_i - T_i^{-1} = (q^{-1}-q)\cdot 1$. It follows that $q^{-n}\varphi_m(b_1T_ib_2) - q^n\varphi_m(b_1T_i^{-1}b_2) =$

$=(q^{-1}-q)\varphi_m(b_1b_2)$ that translates into $q^{-n}P(L_+)-q^nP(L_-)=$ $=(q^{-1}-q)P(L_0)$. Also if L is the unlink w. k components, then $L=\overline{1_k}$, where $1_k\in B_k$ is the unit and $P(L)=tr(K^{\otimes k})=tr(K)^k$ $=\left(\frac{q^n-q^{-n}}{q-q^{-1}}\right)^k$. So for each $a=q^n$ we get a knot invariant, let's denote it by $P_n(L)$

But now we can pick any n. Resolving the crossings in L, we get a function in $q^{\pm n}, q^{-1}$ that can be written as a polynomial in $q^{\pm n}, q^{\pm 1}$, $\frac{q^n-q^{-n}}{q-q^{-1}}$ ~~and does~~ that doesn't depend on n. So we can plug a instead of q^n and get a knot invariant.

(3) a) Consider the subalgebra in $\mathbb{K}$ obtained by localizing R in $[i]_q$ for $i<d$. Denote it by R'. We still can specialize q to a primitive dth root of 1 in R'. Consider the $U_{\mathbb{K}}$-module $L_{\mathbb{K}}(Rq^r)$, $0\le r<d$.
We have the basis u_i as in Thm 1.4 in Lec 13. Let $L_{R'}(Rq^r)$ be the R'-span of these basis elements. The restriction of $\mathbb{C}_\epsilon\otimes_{R'}L_{R'}(Rq^r)$ to u_ϵ is $L(R\epsilon^r)$. The elements $E^{(d)},F^{(d)}$ act on $L_{R'}(Rq^r)$ by 0.
The space $\mathbb{C}_\epsilon\otimes_{R'}L_{R'}(Rq^r)$ is naturally $\dot{U}_\epsilon=\mathbb{C}_\epsilon\otimes_{R'}U_{R'}$-module.

b) Analyzing the action of $\binom{K;0}{d}$ on a tensor product is hard because we don't have a formula for Δ of this element. On the other hand, it's easy to describe the actions of $F^{(d)},E^{(d)}$ on $L(R,r)\otimes Fr^*L(m)$.
when $d\nmid i$ ~~Indeed~~ First of all, since $E^{(i)},F^{(i)}$ act on $Fr^*L(m)$ by 0, we get
$E(v\otimes v')=Ev\otimes v'$, $F(v\otimes v')=Fv\otimes v'$, $K(v\otimes v')=Kv\otimes v'$
By the proof of Lem 1.8 in Lec 15, we have

$$\Delta(E^{(d)})=\sum_{i=0}^{d}q^{i(d-i)}E^{(d-i)}K^i\otimes E^{(i)} \quad \Delta(F^{(d)})=\sum_{i=0}^{d}q^{i(d-i)}F^{(d-i)}\otimes F^{(i)}K^{-i}$$

All the summands but ones for $i=d$ act by 0 on $L(R,r)\otimes Fr^*L(m)$ (we have $E^{(d)}v=0=F^{(d)}v$ and $E^{(i)}v'=F^{(i)}v'$ for $0<i<d$. So $E^{(d)}(v\otimes v')=v\otimes ev'$, $F^{(d)}(v\otimes v')=v\otimes fv'$
Now $K(v_r\otimes v_m)=Kv_r\otimes v_m=\epsilon^r v_r\otimes v_m$. To compute the action of

$\binom{K;0}{\alpha}$ note that $E^{(\alpha)}F^{(\alpha)} = \binom{K;0}{\alpha} + \sum_{i=1}^{\alpha-1} F^{(\alpha-i)}\binom{K;2i-2\alpha}{i}E^{(\alpha-i)}$

The summands under Σ act on $v_{\xi n}$ by 0, so $\binom{K;0}{\alpha} v_{\xi n} = E^{(\alpha)}F^{(\alpha)} v_{\xi n} =$
$= v_r \otimes {}^0ef v_m = v_r \otimes h v_m = m\, v_r \otimes v_m = v_{\xi n}$.

c) This follows because $E(v\otimes v') = (Ev)\otimes v'$; $F(v\otimes v') = Fv \otimes v'$; $E^{(\alpha)} v\otimes v' =$
$= v \otimes ev'$; $F^{(\alpha)} v\otimes v' = v\otimes fv'$. The first factor, $L(\xi, r)$, is irreducible w.r.t E, F, and the second factor $Fr^* L(m) = L(m)$ is irreducible w.r.t e, f.

d) The solution is in several steps:

1) Let A be a Hopf algebra, and M be an A-module. Set $M^A = \{m \in M \mid am = \eta(a)m\}$
It's easy to check that $\mathrm{Hom}_A(M,N) = \mathrm{Hom}(M,N)^A$ (use the antipode axiom)

2) So let M be an irreducible $\dot{U}_\epsilon$-module. Inside, we can find an irreducible u_ϵ-submodule M_0. The irreducible u_ϵ-submodules are precisely $L(\xi, r)$ $\xi \in \{\pm 1\}$, $0 \le r < \alpha$, the classification works precisely as for $\mathfrak{sl}_2(\mathbb{F}_p)$. So we pick $L(\xi, r) \simeq M_0$. Consider $\mathrm{Hom}_{u_\epsilon}(M_0, M) = \mathrm{Hom}(M_0, M)^{u_\epsilon}$ We view M_0 as a $\dot{U}_\epsilon$-module as in (a), so $\mathrm{Hom}(M_0, M)$ is a $\dot{U}_\epsilon$-module.

3) Now for a $\dot{U}_\epsilon$-module N, we have $N^{u_\epsilon} = \{n \in N \mid an = \eta(a)n \ \forall a \in u_\epsilon\}$ is a module over $\dot{U}_\epsilon / (a - \eta(a), a \in u_\epsilon) = \dot{U}_\epsilon / (E, F, K-1) \simeq U(\mathfrak{sl}_2)$. We see that $\mathrm{Hom}_{u_\epsilon}(M_0, M)$ is a $U(\mathfrak{sl}_2)$-module ~~and hence, via the Frobenius pull-back, $\dot{U}_\epsilon$-module~~

4) The natural homomorphism $\mathrm{Hom}_{u_\epsilon}(M_0, M) \otimes M_0 \longrightarrow M$ is that of $\dot{U}_\epsilon$-modules. Besides, it's injective. So it's also surjective and $\mathrm{Hom}_{u_\epsilon}(M_0, M)$ is an irreducible $U(\mathfrak{sl}_2)$-module. We are done by (c)