# HW 2 Solutions

Problem 1: 1) We just need to check that $\Delta([x,y]) = [\Delta(x), \Delta(y)]$. This is straightforward

2) $\Delta(x^p) = \Delta(x)^p = (x\otimes 1 + 1\otimes x)^p = \sum_{i=0}^{p} \binom{p}{i} x^i \otimes x^{p-i} = x^p\otimes 1 + 1 \otimes x^p$

3) We have $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$ by PBW. Moreover, we can define $\underline{\Delta}: S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g})$ similarly to $\Delta$. The homomorphism $\Delta$ preserves filtrations and the associated graded homomorphism is $\underline{\Delta}$. So, it's enough to check that if $u \in S(\mathfrak{g})^m$ is primitive & $m<p$, then $m=1$. Let $\varphi_i(u)$ be defined as ~~the coefficient of~~ follows:

Assume $m>1$

$\underline{\Delta}(u) = u\otimes 1 + \sum_{i=1}^{s} \varphi_i(u) \otimes x_i + \dots$ (here $s = \dim \mathfrak{g}$, $x_1, \dots x_s$ denotes a basis)

Then, when $u$ is a ~~mono~~ monomial, we see that $\varphi_i(u) = \frac{\partial u}{\partial x_i}$. The latter therefore holds for any $u$. If $u$ is primitive, then $\varphi_i(u) = 0$ for all $i$. Since $\deg u = m < p$, this means $u=0$. Contradiction

4) $(x+y)^p - x^p - y^p$ is primitive and lies in $U(\mathfrak{g})^{\leq p-1}$. So $(x+y)^p - x^p - y^p \in \mathfrak{g} = \mathfrak{g}_{1}'$. Under the natural associative algebra homomorphism $\varphi: U(\mathfrak{g}) \to \mathfrak{g}$, we get $\varphi((x+y)^p - x^p - y^p) = (x+y)^{[p]} - x^{[p]} - y^{[p]}$. Hence $(x+y)^p - x^p - y^p = (x+y)^{[p]} - x^{[p]} - y^{[p]}$.

Rem: Let $L$ be a free Lie algebra on ~~$x_1, \dots, x_n$~~ $x, y$. So $U(L)$ is a free associative algebra. With suitable modification, 3) works for $L$. So we see that $(x+y)^p - x^p - y^p = z \in L$. ~~$x,y \in L$~~ It follows that for any associative algebra $A/\mathbb{F}$ and any $a, b \in A$, the difference $(a+b)^p - a^p - b^p$ is a Lie polynomial of $a, b$ independent of $A$ (just plug $a$ and $b$ into $z = z(x,y)$). This gives another proof of the claim of this problem

Problem 2: Note that in all 3 cases $\Delta_\alpha(z)$ admits a weight decomposition $\Delta_\alpha(z) = \bigoplus_{i=0}^{p-1} \Delta_\alpha(z)_{z-i}$. In cases 1 & 2 ($\alpha = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ or $\begin{pmatrix}0&1\\0&0\end{pmatrix}$) we have $z \in \mathbb{F}_p$, while in case 3, $z \notin \mathbb{F}_p$. Moreover, $f\Delta_\alpha(z)_{z-i} = \Delta_\alpha(z)_{z-i-1}$ if $i<p-1$ in all cases while in case 2 we also have $f\Delta_\alpha(z)_{z+1-p} = \Delta_\alpha(z)_z$

~~Case 1~~ We have $ef^i v_z = i(z-i+1)f^{i-1}v_z$ (1)

Claim: If $U \subset \Delta_\alpha(z)$, then $U = \bigoplus_{j\geq i}^{p-1} \Delta_\alpha(z)_{z-j}$ for some $j$ w. $z-i+1=0$

Proof: $U$ is the sum of weight spaces and contains $\Delta_\alpha(z)_{z-i-1}$ with each $\Delta_\alpha(z)_{z-i}$ if $i<p-1$. If $U=\Delta_\alpha(z)$, then the claim is trivially true. Otherwise take minimal $i$ s.t. $\Delta_\alpha(z)_{z-i} \subset U$. $\square$

Case 3: $z-i+1\neq 0$ $\forall i$. So $\Delta_\alpha(z)$ is irreducible. It has a unique vector annihilated by $e$ w. weight $z$. This shows $\Delta_\alpha(z) \not\cong \Delta_\alpha(z')$ if $z\neq z'$.

Case 2. In the claim, $i=0$. So $\Delta_\alpha(z)$ is irreducible. It has one vector annihilated by $e$ if $z=p-1$ and two such vectors else (w. weights $z$ and $-2-z$). This gives a nonzero homomorphism $\Delta_\alpha(-2-z)\longrightarrow\Delta_\alpha(z)$ that has to be iso b/c both modules are irreducible. On the other hand $\Delta_\alpha(z)\not\cong\Delta_\alpha(z')$ if $z'\neq z,-2-z$ for the same reasons

Case 1 Here the only proper submodule $U\subset\Delta_\alpha(z)$ is $L_\alpha(-2-z)$ by (1), it exists if $z\neq p-1$. All $L_\alpha(z)$ are pairwise non-isomorphic.

Problem 3: Since $V$ is rational, $\begin{pmatrix}1&x\\0&1\end{pmatrix}v = \sum_m v_m(x)$, where $v_m(x)\in V_m[x]$.
From $\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}z^{-1}&0\\0&z\end{pmatrix}=\begin{pmatrix}1&z^2x\\0&1\end{pmatrix}$, we deduce that
$z^{m-n}v_m(x)=v_m(z^2x)$. So $v_m(x)=0$ for $m<n$. Also $v_n(x)=v_n(z^2x)$, which implies that $v_n(x)$ is constant. Since $v_n(0)=v$, we see that $\begin{pmatrix}1&x\\0&1\end{pmatrix}v=v$.

Problem 4. 1) Using the invariant form $(x,y)=tr(xy)$ on $\mathfrak{g}$ we identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then $\sum_{i,j=1}^n E_{ij}\otimes E_{ji}\in\mathfrak{g}\otimes\mathfrak{g}$ corresponds to $id\in End(\mathfrak{g})=\mathfrak{g}\otimes\mathfrak{g}^*$. So this element is in $(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$. So the action of $\sum_{i,j=1}^n E_{ij}\otimes E_{ji}$ on $M_1\otimes M_2$ commutes with $\mathfrak{g}$. There's also a direct solution.

2) The endomorphisms $T_1,\dots,T_{n-1}$ satisfy the relations in $S_n$.
Let's prove that $X_iX_j=X_jX_i$ for $i<j$. Let $N=M\otimes V^{j-1}$. Then $X_i$ is an endomorphism of this $\mathfrak{g}$-module. So for $v\in V$, $n\in M\otimes V^{j-1}$ we have

$$X_iX_j(v\otimes n)=\sum_{l,j=1}^n E_{lj}v\otimes E_{jl}X_in=[\text{part (1)}]=\sum_{l,j=1}^n E_{lj}v\otimes X_iE_{jl}n = X_jX_i(v\otimes n)$$

It's clear that $X_iT_j=T_jX_i$ for $|i-j|>1$. In order to prove that $T_iX_{i+1}=X_iT_i+1$ it's enough to assume that $i=1$. Let $u,v\in V$, $m\in M$.

$$X_1T_1(v\otimes u\otimes m)=X_1(u\otimes v\otimes m)=\sum_{i,j=1}^n u\otimes E_{ij}v\otimes E_{ji}m$$

$$T_1X_2(v\otimes u\otimes m)=T_1\sum_{i,j=1}^n E_{ij}v\otimes E_{ji}(u\otimes m)=T_1\sum_{i,j=1}^n E_{ij}v\otimes(E_{ji}u\otimes m+u\otimes E_{ji}m)=\sum_{i,j=1}^n E_{ji}u\otimes E_{ij}v\otimes m+\underbrace{\sum_{i,j=1}^n u\otimes E_{ij}v\otimes E_{ji}m}_{=X_1T_1(v\otimes u\otimes m)}$$

It remains to prove that $\sum_{i,j=1}^n E_{ij}u\otimes E_{ji}v=v\otimes u$. This is checked directly on basis elements $e_k\otimes e_l$.

Problem 5: 1) $[\cdot,\cdot]$ is clearly skew-symmetric. Let's check Jacobi identity

$\cancel{[[x,y],z]}$ $[[x\otimes t^k, y\otimes t^l], z\otimes t^m] = [c \text{ is central}] = [[x,y]\otimes t^{k+l}, z\otimes t^m]$

$= [[x,y],z]\otimes t^{k+l+m} + (k+l)\delta_{k+l+m,0}\, tr([x,y]z)c$

So the Jacobi id. will follow if we check that, for $k+l+m=0$:

$(k+l)tr([x,y],z) + (l+m)tr([y,z]x) + (m+k)tr([z,x]y) = 0$

The l.h.s. is $((k+l)tr(xyz) + (l+m)tr(yzx) + (m+k)tr(zxy)) -$

$-((k+l)tr(yxz) + (l+m)tr(zyx) + (m+k)tr(xzy))$. We have $tr(xyz) =$

$= tr(yzx) = tr(zxy)$ and $tr(yxz) = tr(zyx) = tr(xzy)$. So both brackets above are zero as $k+l+m=0$.

2) Relations involving elements $e_i, h_i, f_i$ with $i\neq 0$ only follow from the Serre relations for $\mathfrak{sl}_n$. So we only need to check relations involving $h_0, e_0, f_0$: $[h_0, h_i] = 0$ is obvious

$[e_0, f_0] = [tE_{n,1}, t^{-1}E_{1,n}] = E_{n,n} - E_{1,1} + tr(E_{n1}E_{1n})c = E_{n,n} - E_{1,1} + c = h_0$

All other relations do not have summands of $c$ and can be checked in $\mathfrak{sl}_n[t^{\pm 1}]$. They are all homogeneous in $t$ so we can check them by replacing $h_0$ w. $E_{nn} - E_{11}$, $e_0$ w. $E_{n1}$, $f_0$ w. $E_{1n}$. They now follow from the Serre relations for $\mathfrak{sl}_n$ by shifting the basis in $\mathbb{C}^n$.

3) Set $\delta = \sum_{i=0}^{n-1} \alpha_i$. Note that $s_i(\alpha_j) = \begin{cases} -\alpha_j, & i=j \\ \alpha_i + \alpha_j, & i = j\pm 1 \\ \alpha_j, & \text{else} \end{cases}$

where we view indices as element of $\mathbb{Z}/n\mathbb{Z}$. It follows that $s_i(\delta) = \delta$ $\forall i$. Let $\tilde{Q}$ denote the free abel. group w. basis $\alpha_0, \ldots \alpha_{n-1}$. The group $W$ acts on $\tilde{Q}$. On $\tilde{Q}/\mathbb{Z}\delta$ the action becomes that of the Weyl group of $\mathfrak{sl}_n$, i.e. $S_n$. So we get an epimorphism $W \twoheadrightarrow S_n$. If $\tau$ is an element in the kernel, then $\tau(\alpha_i) = \alpha_i + n_i(\tau)\delta$. Let us produce an element in the kernel. Let $\tilde{\alpha} = \delta - \alpha_0$. Set $\tau = s_0 s_{\tilde{\alpha}}$. For $v \in \tilde{Q}$ we get

$\tau(v) = s_0 s_{\tilde{\alpha}}(v) = s_0(v - (v,\tilde{\alpha})\tilde{\alpha}) = s_0(v + (v,\alpha_0)(\delta - \alpha_0)) =$

$= v - (\alpha_0, v)\alpha_0 + (v,\alpha_0)\delta - (v,\alpha_0)s_0(\alpha_0) = v + (v,\alpha_0)\delta = v - (v,\tilde{\alpha})\delta$

For $\lambda \in Q$, let us write $\tau_\lambda$ for $\tau_\lambda(v) = v - (v,\lambda)\delta$. Then $\tau_{\lambda_1+\lambda_2} = \tau_{\lambda_1}\circ\tau_{\lambda_2}$

and $\sigma \tau_\lambda \sigma^{-1} = \tau_{\sigma(\lambda)}$ for $\sigma \in S_n$. From here we deduce that we have a homomorphism $S_n \ltimes Q \to W$ that sends $\sigma \in S_n$ to $\sigma \in S_n \subset W$ and $\lambda \in Q$ to $\tau_\lambda$. This homomorphism is surjective because $S_n$ and $\tau_{\tilde{\alpha}} = s_0 s_{\tilde{\alpha}}$ generate $W$. Its kernel is a normal subgroup in $Q$ stable under $S_n$. If it is nonzero, then it has full rank and hence $S_n \ltimes Q/\ker$ is finite. But $\langle \tau_{\tilde{\alpha}} \rangle \in W$ is infinite. Contradiction. So $S_n \ltimes Q \xrightarrow{\sim} W$. $Q$ is precisely $\{(x_1 \ldots x_n) \in \mathbb{Z}^n \mid \sum x_i = 0\}$ $\square$

4) $\Delta^{re} = W\{\alpha_0, \ldots \alpha_n\} = \{\alpha + n\delta, \ \overbrace{\alpha \in \Delta(\mathfrak{sl}_n)}^{n \in \mathbb{Z}}\}$. Now note that $n\delta$ is a root for $\hat{\mathfrak{sl}}_n$ (w. $\mathfrak{g}_{n\delta} = t^n \mathfrak{h}$). So it is a root for $\mathfrak{g}(A)$. The radical of the invariant form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ is spanned by $\delta$. So $\Delta^{im} = \{n\delta, \ n \in \mathbb{Z} \setminus \{0\}\}$.

5) Let $\mathfrak{k}$ denote the kernel of $\mathfrak{g}(A) \twoheadrightarrow \hat{\mathfrak{sl}}_n$. We see that $\Delta(\mathfrak{g}(A)) = \Delta(\hat{\mathfrak{sl}}_n)$. So $\mathfrak{k} \subset \bigoplus_{n \neq 0} \mathfrak{g}_{n\delta}$. Suppose $x \in \mathfrak{k} \cap \mathfrak{g}_{n\delta}^{(n>0)}$. Then $e_\alpha x = f_\alpha x = 0 \ \forall \alpha \in \Delta^{re}$. On the other hand, $x = \sum e_i y_i$ w. $y_i \in \mathfrak{g}_{n\delta - \alpha_i}$. Note that $n\delta - \alpha_i + \alpha_j$ is not a root for $i \neq j$. So $0 = f_e x = \sum f_e e_i y_i = f_e e_e y_e$. But $f_e e_e y_e = 0 \Rightarrow e_e y_e = 0$. So $x = 0$ and we are done.