LECTURE 5: SEMISIMPLE LIE ALGEBRAS OVER $\mathbb{C}$

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Introduction

In this lecture I will explain the classification of finite dimensional semisimple Lie algebras over $\mathbb{C}$. Semisimple Lie algebras are defined similarly to semisimple finite dimensional associative algebras but are far more interesting and rich. The classification reduces to that of simple Lie algebras (i.e., Lie algebras with non-zero bracket and no proper ideals). The classification (initially due to Cartan and Killing) is basically in three steps.

1) Using the structure theory of simple Lie algebras, produce a combinatorial datum, the root system.

2) Study root systems combinatorially arriving at equivalent data (Cartan matrix/ Dynkin diagram).

3) Given a Cartan matrix, produce a simple Lie algebra by generators and relations.

In this lecture, we will cover the first two steps. The third step will be carried in Lecture 6.

1. Semisimple Lie algebras

Our base field is $\mathbb{C}$ (we could use an arbitrary algebraically closed field of characteristic 0).

1.1. Criteria for semisimplicity.

We are going to define the notion of a semisimple Lie algebra and give some criteria for semisimplicity. This turns out to be very similar to the case of semisimple associative algebras (although the proofs are much harder).

Let $\mathfrak{g}$ be a finite dimensional Lie algebra.

Definition 1.1. We say that $\mathfrak{g}$ is *simple*, if $\mathfrak{g}$ has no proper ideals and $\dim \mathfrak{g} > 1$ (so we exclude the one-dimensional abelian Lie algebra). We say that $\mathfrak{g}$ is *semisimple* if it is the direct sum of simple algebras.

Any semisimple algebra $\mathfrak{g}$ is the Lie algebra of an algebraic group, we can take the automorphism group $\mathrm{Aut}(\mathfrak{g})$. The connected component of 1 is denoted by $\mathrm{Ad}(\mathfrak{g})$, it should be viewed as the group of "inner" automorphisms of $\mathfrak{g}$. One can show that the algebra $\mathfrak{g}$ is simple if and only if $\mathrm{Ad}(\mathfrak{g})$ is simple as an abstract group.

We define the *Killing form* on a finite dimensional Lie algebra $\mathfrak{g}$ by $(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$, this is a symmetric bilinear form. It is *invariant* in the sense that

$$([x, y], z) + (y, [x, z]) = 0 \tag{1.1}$$

(equivalently, $(\cdot,\cdot)$ is annihilated by the representation of $\mathfrak{g}$ in the space $S^2(\mathfrak{g}^*)$). If $\mathfrak{g}$ is a Lie algebra of an algebraic group (or complex Lie group) G, then $(\cdot,\cdot)$ is G-invariant in the usual sense $(g\cdot y, g\cdot z) = (y, z)$ (just differentiate the latter equality to get (1.1)).

Theorem 1.2. *The following conditions are equivalent:*

(1) $\mathfrak{g}$ *is semisimple.*

(2) $(\cdot,\cdot)$ *is non-degenerate.*
(3) *Any finite dimensional representation of* $\mathfrak{g}$ *is completely reducible.*

We also have the notion of the radical of $\mathfrak{g}$ (=the maximal solvable ideal). The algebra $\mathfrak{g}$ is semisimple if and only if the radical is zero.

(2) gives a practical way to check semisimplicity, for example, $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ (for $n > 2$) and $\mathfrak{sp}_n(\mathbb{C})$ are semisimple, this is left as an exercise.

1.2. **Cartan subalgebras.** Let $\mathfrak{g}$ be a semisimple Lie algebra. We say that an element in $\mathfrak{g}$ is semisimple (resp., nilpotent) if it acts by a semisimple (resp., nilpotent) operator in some faithful finite dimensional representation. Then it acts by a semisimple (resp., nilpotent) operator in any finite dimensional representation.

Proposition 1.3. *The following is true:*

(1) *A Zariski generic element* $x \in \mathfrak{g}$ *is semisimple.*
(2) *The centralizer* $\mathfrak{z}_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} | [x, y] = 0\}$ *is an abelian subalgebra in* $\mathfrak{g}$ *consisting of semisimple elements. This centralizer is called a* Cartan subalgebra.
(3) *Any two Cartan subalgebras are conjugate by an element of* $\mathrm{Ad}(\mathfrak{g})$.

For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we take x with all distinct eigenvalues. Any Cartan subalgebra is the subalgebra of all elements diagonal in some basis. In the definition of $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$, we use the form $(u, v) = \sum_{i=1}^{n} u_{n+1-i}v_i$ so that $\mathfrak{so}_n(\mathbb{C})$ consists of all matrices that are skew-symmetric with respect to the main antidiagonal. We again take x with all distinct eigenvalues. For a Cartan subalgebra, we can take the subalgebra of all diagonal matrices contained in $\mathfrak{so}_n(\mathbb{C})$, these matrices have the form $(x_1, \ldots, x_m, -x_m, \ldots, -x_1)$ if $n = 2m$ and $(x_1, \ldots, x_m, 0, -x_m, \ldots, -x_1)$ if $n = 2m + 1$. The case $\mathfrak{g} = \mathfrak{sp}_{2m}(\mathbb{C})$ is treated similarly – we take the form $\omega(x, y) = \sum_{i=1}^{m}(x_i y_{2n+1-i} - x_{2n+1-i} y_i)$.

1.3. **Root systems.** Let $\mathfrak{h}$ denote a Cartan subalgebra. For $\alpha \in \mathfrak{h}^*$, we set $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$. The set of all $\alpha \in \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq \{0\}$ is called the *root system* of $\mathfrak{g}$. We will write Δ (or $\Delta(\mathfrak{g})$) for the root system so that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$. Note that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Note also that $(\cdot,\cdot)$ restricts to a perfect pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ (indeed, $\mathfrak{g}_\alpha$ is orthogonal to any $\mathfrak{g}_\beta$ with $\alpha + \beta \neq 0$) and to a non-degenerate form on $\mathfrak{h}$. So we have a non-degenerate symmetric form on $\mathfrak{h}^*$ also denoted by $(\cdot,\cdot)$.

We have the following properties of the subspaces $\mathfrak{g}_\alpha$. (1)-(6) are obtained using the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

Proposition 1.4. *We have the following properties of* $\mathfrak{g}_\alpha$*'s and* Δ.

(1) *Let* $e \in \mathfrak{g}_\alpha$ *and* $f \in \mathfrak{g}_{-\alpha}$ *be such that* $(e, f) \neq 0$. *Then we can rescale* f *in such a way that* $h := [e, f]$ *(that is an element of* $\mathfrak{h}$*) satisfies* $\alpha(h) = 2$. *We have* $[h, e] = 2e, [h, f] = -2f$. *In other words, the map* $\mathfrak{sl}_2 \to \mathfrak{g}$ *given by* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h$ *is a Lie algebra homomorphism.*
(2) $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Delta$. *So we have elements* $e_\alpha \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}, h_\alpha \in \mathfrak{h}$ *as in (1). Note that* h_α *is uniquely determined.*
(3) $\beta(h_\alpha) \in \mathbb{Z}$ *for any* $\alpha, \beta \in \Delta$.
(4) *If* $\alpha \in \Delta$, *then* $2\alpha \notin \Delta$.
(5) *For* $\beta \in \Delta$, *define a linear map* $s_\beta : \mathfrak{h}^* \to \mathfrak{h}^*$ *by* $\lambda \mapsto \lambda - \lambda(h_\beta)\beta$. *The map* s_β *maps* Δ *to itself.*

(6) *Let* $\alpha+\beta\neq 0$*. If* $\beta+k\alpha,\beta+\ell\alpha\in\Delta$*, then* $\beta+i\alpha\in\Delta$ *for any integer* i *between* k,ℓ*.*
(7) Δ *spans* $\mathfrak{h}^*$*. Moreover, let* $\mathfrak{h}_{\mathbb{R}}$ *denotes the* $\mathbb{R}$*-span of* Δ*. Then the restriction of* $(\cdot,\cdot)$ *to* $\mathfrak{h}_{\mathbb{R}}$ *is positive definite.*
(8) *Under the identification of* $\mathfrak{h}^*$ *and* $\mathfrak{h}$ *by means of* $(\cdot,\cdot)$*, we get* $h_\alpha=2\alpha/(\alpha,\alpha)$ *(we'll use notation* $\alpha^\vee$ *for the right hand side).*

Example 1.5. Let $\mathfrak{g}=\mathfrak{sl}_n$. Let ϵ_i denote the linear function on $\mathfrak{h}$ taking the entry (i,i) so that $\sum_{i=1}^n\epsilon_i=0$. Then the root system Δ consists of the elements $\epsilon_i-\epsilon_j, i\neq j$. The root subspace $\mathfrak{g}_\alpha$ for $\alpha=\epsilon_i-\epsilon_j$ is spanned by the unit matrix E_{ij}. The root system Δ is called the root system of type A_{n-1}.

Note that, for $x\in\mathfrak{h}$, we have $(x,x)=\sum_{\alpha\in\Delta}\alpha(x)^2=2\sum_{i<j}(x_i-x_j)^2=2(n-1)\sum_{i=1}^n x_i^2-4\sum_{i<j}x_ix_j$. Since $\sum_{i=1}^n x_i=0$, we arrive at $(x,x)=2(n+1)\sum_{i=1}^n x_i^2$.

Example 1.6. Let $\mathfrak{g}=\mathfrak{so}_{2n+1}$. Then $\mathfrak{h}^*$ has a basis of functions $\epsilon_i, i=1,\ldots,n$ defined as in the previous example. The root system Δ consists of the elements $\pm\epsilon_i\pm\epsilon_j, i\neq j,\pm\epsilon_i$. This is the root system of type B_n.

Example 1.7. For $\mathfrak{g}=\mathfrak{sp}_{2n}$, the root system Δ consists of $\pm\epsilon_i\pm\epsilon_j, i\neq j,\pm 2\epsilon_i$ (type C_n).

Example 1.8. For $\mathfrak{g}=\mathfrak{so}_{2n}$, the root system Δ consists of $\pm\epsilon_i\pm\epsilon_j, i\neq j$, (type D_n).

In the last three examples, the form (x,x) on $\mathfrak{h}$ is proportional to $\sum_{i=1}^n x_i^2$.

1.4. **Irreducible root systems.** We say that Δ is *irreducible* if there are no proper subspaces $\Delta_1,\Delta_2\subset\Delta$ such that $\Delta=\Delta_1\cup\Delta_2$ and $(\alpha_1,\alpha_2)=0$ for $\alpha_i\in\Delta_i$.

Lemma 1.9. *The algebra* $\mathfrak{g}$ *is simple if and only if* Δ *is irreducible.*

Example 1.10. The algebras $\mathfrak{sl}_n,\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n}$ are simple for $n>1$. The algebra $\mathfrak{so}_{2n}$ is simple if and only if $n>2$. For $n=2$, the root system Δ equals $\{\pm\epsilon_1\pm\epsilon_2\}$ and we can take $\Delta_1=\{\pm(\epsilon_1-\epsilon_2)\}$ and $\Delta_2=\{\pm(\epsilon_1+\epsilon_2)\}$. And, indeed, we have $\mathfrak{so}_4\cong\mathfrak{sl}_2\times\mathfrak{sl}_2$.

2. Classification of root systems

2.1. **Abstract root systems.** Let E be a finite dimensional Euclidian space and $\Delta\subset E\setminus\{0\}$ be a finite collection of elements. For $\alpha\in\Delta$, we write $\alpha^\vee$ for $\frac{2\alpha}{(\alpha,\alpha)}$. Suppose that the following is true

(R1) $(\alpha^\vee,\beta)$ is an integer for any $\alpha,\beta\in\Delta$.
(R2) Define an automorphism s_β of E given by $s_\beta(v)=v-(\beta^\vee,v)\beta$. This automorphism preserves Δ.
(R3) Δ spans E.

Note that $s_\alpha(\alpha)=-\alpha$ and so Δ is closed under multiplication by -1.

We say that Δ is *reduced* if $\alpha\in\Delta$ implies $2\alpha\not\in\Delta$. We see that $\Delta(\mathfrak{g})$ is a reduced root system in $E=\mathfrak{h}^*_{\mathbb{R}}$. Similarly to Section 1.4, we can speak about irreducible root systems.

We say that two root systems $\Delta\subset E,\Delta'\subset E'$ are *isomorphic* if there is a linear isomorphism $\varphi:E\to E'$ such that $\varphi(\Delta)=\Delta'$ and $(\alpha^\vee,\beta)=(\varphi(\alpha)^\vee,\varphi(\beta)),\forall\alpha,\beta\in\Delta$.

2.2. **Weyl group and Weyl chambers.** Note that s_β is the reflection about the hyperplane $\beta^\perp\subset E$. We consider the subgroup $W\subset \mathrm{O}(E)$ generated by the reflections s_β. It preserves Δ. Since Δ is finite and spans E, we see that W is finite. It is called the *Weyl group* of Δ.

Example 2.1. Let Δ be the root system of type A_n. We can take $(x,x)=\sum_{i=1}^{n+1} x_i^2$ to be the scalar product on E. Let $\beta=\epsilon_i-\epsilon_j$. Then $\beta^\vee=\epsilon_i-\epsilon_j$ and $(\beta^\vee,x)=x_i-x_j$, so the reflection s_β is given by $x\mapsto x-(x_i-x_j)(\epsilon_i-\epsilon_j)$, where $x=\sum_{i=1}^{n+1}x_i\epsilon_i$. So it just swaps the ith and jth coordinates of x. It follows that the Weyl group is the symmetric group S_{n+1}. In types B_n, C_n, we get the group $S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$, where the elements of S_n permute the basis vectors $\epsilon_1,\ldots,\epsilon_n\in E$ and the elements of $(\mathbb{Z}/2\mathbb{Z})^n$ switch the signs of the basis vectors. In type D_n, the group $W(D_n)$ is the index two normal subgroup of $S_n\times\mathbb{Z}/2\mathbb{Z}$ consisting of all elements that switch an even number of signs.

The hyperplanes $\beta^\perp$ split E into the union of regions called *Weyl chambers*. The following proposition describes the properties of the chambers.

Proposition 2.2. *The following is true.*

(1) *The group W permutes the Weyl chambers simply transitively.*
(2) *For each Weyl chamber C, there are n roots $\alpha_1,\ldots,\alpha_n$ such that $C=\{v\in E|(\alpha_i,v)\geqslant 0, i=1,\ldots,n\}$.*
(3) *C is a* fundamental domain *for W meaning that for each $v\in E$, there is a unique $u\in C$ with $v\in Wu$.*
(4) *The reflections $s_{\alpha_1},\ldots,s_{\alpha_n}$ (a.k.a. simple reflections) generate the Weyl group.*

Example 2.3. Consider the root system of type A_n. The hyperplanes $\beta^\perp$ are $x_i=x_j$. So we have $(n+1)!$ Weyl chambers, they are specified by an ordering of $x_1,\ldots,x_{n+1}$. An example of a Weyl chamber is given by $C=\{(x_1,\ldots,x_{n+1})|x_1\geqslant x_2\geqslant\ldots\geqslant x_{n+1}\}$. The corresponding simple reflections are the simple transpositions $(i,i+1), i=1,\ldots,n$. They clearly generate S_n.

In types B_n, C_n, we can take $C=\{(x_1,\ldots,x_n)|x_1\geqslant x_2\geqslant\ldots\geqslant x_n\geqslant 0\}$. In type D_n, we can take $C=\{(x_1,\ldots,x_n)|x_1\geqslant\ldots\geqslant x_n\geqslant -x_{n-1}\}$.

2.3. **Simple roots, Cartan matrix and Dynkin diagram.** By a system of simple roots, we mean a subset of the form $\alpha_1,\ldots,\alpha_n$ for some Weyl chamber C, see (2) of Proposition 2.2. Note that W acts simply transitively on the set of systems of simple roots. So it does not matter which system we pick.

Proposition 2.4. *Let $\alpha_1,\ldots,\alpha_n$ be a system of simple roots. Then*

(1) *$\alpha_1,\ldots,\alpha_n$ form a basis in E.*
(2) *For any $\beta\in\Delta$, either $\beta=\sum_{i=1}^n n_i\alpha_i$ with all $n_i\geqslant 0$ (positive root), or $\beta=\sum_{i=1}^n n_i\alpha_i$ with all $n_i\leqslant 0$ (negative root).*

Example 2.5. For the chambers chosen in Example 2.3, we have the following systems of simple roots:

(A_n) $\alpha_i:=\epsilon_i-\epsilon_{i+1}, i=1,\ldots,n$.
(B_n) $\alpha_i:=\epsilon_i-\epsilon_{i+1}, i=1,\ldots,n-1, \alpha_n:=\epsilon_n$.
(C_n) $\alpha_i:=\epsilon_i-\epsilon_{i+1}, i=1,\ldots,n-1, \alpha_n:=2\epsilon_n$.
(D_n) $\alpha_i:=\epsilon_i-\epsilon_{i+1}, i=1,\ldots,n-1, \alpha_n:=\epsilon_{n-1}+\epsilon_n$.

We can encode a simple root system by the *Cartan matrix*, this is an $n\times n$-matrix with entries $n_{ij}=\alpha_i^\vee(\alpha_j)$. This matrix is defined up to a conjugation with a monomial matrix (corresponding to re-ordering $\alpha_1,\ldots,\alpha_n$). We have the following important result.

Proposition 2.6. *Let Δ,Δ' be two reduced root systems with the same Cartan matrix. Then Δ,Δ' are isomorphic.*

We can depict Cartan matrices (or simple root systems) getting so called Dynkin diagrams. Namely, the simple roots are nods. We draw $n_{ij}n_{ji} = \frac{4(\alpha_i,\alpha_j)^2}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}$ un-oriented edges between two nodes. We orient them by putting an arrow (that also can be viewed as an inequality sign) from the longer root to a shorter root. Note that, since $n_{ij}n_{ji} < 4$, we can recover the lengthes of roots from that. The Dynkin diagrams for the root systems A_n-D_n are shown in Picture 1.

2.4. **Classification of Cartan matrices.** By an abstract Cartan matrix we mean a square matrix $A = (a_{ij})_{i,j=1}^n$ such that

(1) $a_{ii} = 2$,
(2) $a_{ij} \leqslant 0$ for $i \neq j$,
(3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We say that A is *irreducible* if we cannot partition $\{1, 2, \ldots, n\}$ into $I \sqcup J$ such that $a_{ij} = 0, i \in I, j \in J$. We say that A is *symmetrizable* if there is a diagonal matrix D with positive entries and a symmetric matrix G such that $A = DG$. Note that if A is irreducible, then D is defined up to a positive scalar factor. We say that A is *positive definite* if so is S.

The matrix $A(\Delta)$ is symmetrizable, we can take $D = \mathrm{diag}(2/(\alpha_i, \alpha_i))_{i=1}^n$, then S is the Gram matrix of the basis α_i. So A is positive definite.

One can classify the irreducible symmetrizable positive definite Cartan matrices. Besides the matrices/ Dynkin diagrams A_n-D_n, there are just five more exceptional diagrams, E_6, E_7, E_8, F_4, G_2, see Picture 2.

One can explicitly produce the root systems corresponding to these diagrams. We are not going to do that. Instead, we will produce simple Lie algebras with these Dynkin diagrams. This will complete the classification of simple Lie algebras (over $\mathbb{C}$ or over general algebraically closed fields of characteristic 0).