LECTURE 16: REPRESENTATIONS OF QUIVERS

IVAN LOSEV

Introduction

Now we proceed to study representations of quivers. We start by recalling some basic definitions and constructions such as the path algebra and indecomposable representations. Then we state a theorem of Kac that describes the dimensions, where the indecomposable representations occur as well as the number of parameters needed to describe their isomorphism classes. We will prove the Kac theorem only partially using Crawley-Boevey's approach based on deformed preprojective algebras. This approach does not allow to prove Kac's theorem completely but it is more elementary than Kac's original approach.

1. Representations of quivers

1.1. **Quivers and their representations.** By a *quiver* we mean an oriented graph, possibly with multiple edges and loops. Formally, it is a quadruple $Q := (Q_0, Q_1, t, h)$, where Q_0, Q_1 are finite sets of vertices and arrows and $t, h : Q_1 \to Q_0$ are maps (taking an arrow a to its tail and head), see Picture 1.1. See Pictures 1.2, 1.3 for some examples of quivers.

A representation of a quiver Q is an assignment that takes every vertex $i \in Q_0$ to a vector space V_i and every arrow $a \in Q_1$ to a map $x_a : V_{t(a)} \to V_{h(a)}$. In particular, a representation of the quiver in Picture 1.2(a) is a pair V_1, V_2 of vector spaces and maps $x_a : V_1 \to V_2, x_b : V_2 \to V_1$.

As with groups and algebras, a representation of a quiver Q is the same thing as a module over a suitable associative algebra, the *path algebra* $\mathbb{C}Q$ of Q. This algebra is constructed as follows. A basis in this algebra is formed by all paths in Q. By a path in Q we mean either the empty path $p = \epsilon_i, i \in Q_0$, or a sequence $p = (a_1, \ldots, a_k)$ of arrows with $t(a_i) = h(a_{i+1})$. We set $t(p) = t(a_k), h(p) = h(a_1)$ (and $h(\epsilon_i) = t(\epsilon_i) = i$). The multiplication is introduced as follows: we have $p_1p_2 = 0$ if $h(p_2) \neq t(p_1)$, and p_1p_2 is the concatenation of p_1, p_2 else. Note that $\mathbb{C}Q$ becomes an associative algebra with unit $1 = \sum_{i\in Q_0} \epsilon_i$.

Let us produce a natural bijection between the representations of Q and the $\mathbb{C}Q$-modules. Given a representation (V_i, x_a) we define the $\mathbb{C}Q$-module $V := \bigoplus_{i\in Q_0} V_i$, where the action is introduced by $\epsilon_i u_j = \delta_{ij}u_j, au_j = x_a(\delta_{jt(a)}u_j)$, where $u_j \in V_j$. We extend the multiplication to an arbitrary path p in an obvious way. Conversely, given a $\mathbb{C}Q$-module V, we get a representation (V_i, x_a) by $V_i := \epsilon_i V, x_a = a\epsilon_{t(a)}$.

In particular, we see that the representations of Q form an abelian category. A morphism $(V_i, x_a) \to (V_i', x_a')$ can be interpreted as a collection of maps $y_i : V_i \to V_i'$ with $y_{h(a)} \circ x_a = x_a' \circ y_{t(a)}$.

We are interested in the case when $\dim V_i < \infty$ for all i. Then we can define the dimension vector $v := (\dim V_i)_{i\in Q_0}$.

1.2. **Indecomposable representations and equivalence.** As usual, the main goal in our study of the representations of Q (equivalently, of $\mathbb{C}Q$) is their classification up to an isomorphism. The Krull-Schmidt theorem reduces this task to the classification of the indecomposable representations. Recall that by an indecomposable representation of an algebra A we mean a representation U that does not split into the direct sum of two nonzero representations of A. Clearly, any finite dimensional representation decomposes into the sum of indecomposable ones. The Krull-Schmidt theorem says that such a decomposition is unique. More precisely, we have the following.

Theorem 1.1. *Let A be an algebra, U be a finite dimensional A-module and $U = \bigoplus_{i\in I} U_i = \bigoplus_{j\in J} U'_j$ be two decompositions into the indecomposable representations. Then there is a bijection $\sigma : I \xrightarrow{\sim} J$ such that $U_i \cong U'_{\sigma(i)}$.*

So what we want to study is the indecomposable representations of $\mathbb{C}Q$ up to an isomorphism. What makes this problem especially nice is that it reduces to describing orbits of a reductive group action on a vector space. Namely, fix vector spaces V_i, let v be the dimension vector. Then the linear maps (x_a) naturally form a vector space $\operatorname{Rep}(Q,v) = \bigoplus_{a\in Q_1} \operatorname{Hom}(V_{t(a)}, V_{h(a)})$. On this space we have an action of the group $G_v := \prod_{i\in Q_0} \operatorname{GL}(V_i)$ given by "change of bases": $(g_i).(x_a) = (g_{h(a)}x_a g_{t(a)}^{-1})$. Clearly, the representations of $\mathbb{C}Q$ on V given by $(x_a), (x'_a)$ are isomorphic if and only if (x_a) and (x'_a) lie in the same G_v-orbit. So what we need to do is to describe the G_v-orbits in $\operatorname{Rep}(Q,v)$.

To finish this section let us provide two classical linear algebraic examples.

Example 1.2. Let us consider the quiver of type A_2 (two vertices and one arrow between them). Then $\operatorname{Rep}(Q,v) = \operatorname{Hom}(V_1,V_2)$ and $G_v = \operatorname{GL}(V_1)\times \operatorname{GL}(V_2)$ acts on this space $(g_1,g_2).x = g_2xg^{-1}$. The maps x,x' lie in the same orbit if and only if $\operatorname{rk} x = \operatorname{rk} x'$. Such a map is indecomposable if and only if $\dim V_1 = 1, \dim V_2 = 0$ or vice versa (and x_a is zero) or $\dim V_1 = \dim V_2 = 1$ and x_a is an isomorphism.

Example 1.3. Consider the case when Q is the Jordan quiver. Here V is a single vector space and $G_v = \operatorname{GL}(V)$ acts on V by conjugations. The orbits are classified by the Jordan normal forms (up to permutation of blocks). The indecomposable representations are the single Jordan blocks.

1.3. **Questions.** In general, the complete description of the G_v-orbits in $\operatorname{Rep}(Q,v)$ is a wild problem that can only be solved completely when Q is a type A,D,E or a type $\tilde{A},\tilde{D},\tilde{E}$ type diagram (see Picture 1.3. for the latter).

One question that we will answer completely is about the possible dimensions of the indecomposable representations. We will also describe the "number of parameters" that is needed to describe the equivalence classes of the indecomposable representations in $\operatorname{Rep}(Q,v)$. Let us make this formal.

Let X be an algebraic variety equipped with an action of an algebraic group G. Define the subset $X_{\leqslant i} := \{x\in X \mid \dim Gx \leqslant i\}$ and $X_i := X_{\leqslant i}\setminus X_{\leqslant i-1}$. Note that $X_{\leqslant i}$ is a closed subvariety of X and hence X_i is open in $X_{\leqslant i}$. We define $p_G(X)$, the number of parameters for the G-orbits in X, to be $p_G(X) := \max_i(\dim X_i - i)$. Note that $p_G(X)=0$ if and only if X consists of finitely many orbits.

To talk about the number of parameters of the indecomposable representations we need to extend the function p_G to G-stable constructible subsets of X. Recall that a constructible subset, by definition, is a union of locally closed subvarieties. By the Chevalley theorem, the image of a morphism of algebraic varieties is constructible.

Lemma 1.4. *The following is true.*

(1) *A G-stable constructible subset Y is a union of G-stable locally closed subvarieties.*

(2) *The subset of indecomposable representations in* $\mathrm{Rep}(Q,v)$ *is constructible.*

Proof. There is an open subset $Y^0 \subset Y$ such that $\dim Y^0 > \dim \overline{Y \setminus Y^0}$. We can replace Y^0 with GY^0 (still open) and assume that Y^0 is G-stable. Then we induct on $\dim \overline{Y}$ to prove (1).

Let us prove (2). For any decomposition $v = v' \oplus v''$, we have a G_v-equivariant morphism $\psi_{v',v''} : G_v \times \mathrm{Rep}(Q,v') \times \mathrm{Rep}(Q,v'') \to \mathrm{Rep}(Q,v)$, $(g,(x'_a),(x''_a)) \mapsto g.(x'_a \oplus x''_a)$. The set of the indecomposable representations is the complement to $\bigcup \mathrm{im}\,\psi_{v',v''}$, where the union is taken over all proper decompositions $v = v' \oplus v''$. Hence it is constructible. ☐

This lemma allows to define p_v, the number of parameters needed to describe the orbits of indecomposable representations in $\mathrm{Rep}(Q,v)$.

Example 1.5. For the quiver of Dynkin type A_2, the number of orbits is finite. So the number of parameters is 0.

Example 1.6. For the Jordan quiver, $p_n = 1$, for all $n \in \mathbb{Z}_{>0}$.

2. Kac's theorems

2.1. **Root system and Weyl group.** It turns out that the answers to our questions about $\mathrm{Rep}(Q,v)$ are stated in terms of the root system of the Kac-Moody algebra $\mathfrak{g}(A)$, where A is the Cartan matrix of Q, given by $a_{ij} = 2\delta_{ij} - n_{ij}$, where n_{ij} stands for the number of edges between i,j. We are going to recall the corresponding definitions in the form that we need (we need to work in a more general setting, where we have loops, in which case the algebra $\mathfrak{g}(A)$ was not defined).

Define two spaces $\mathfrak{h},\mathfrak{h}^*$ with bases $\alpha_i^\vee,\alpha_i$, where $i \in Q_0$. Define the *Tits form* on $\mathfrak{h}^*$ with matrix A. In other words, if we identify $\mathfrak{h}^*$ with $\mathbb{C}^{Q_0}$ by $\sum_{i\in Q_0} x_i\alpha_i \mapsto x = (x_i)_{i\in Q_0}$, then the form is given by $(x,y) = 2\sum_{i\in Q_0} x_iy_i - \sum_{a\in Q_1}(x_{t(a)}y_{h(a)} - y_{t(a)}x_{h(a)})$. For $i \in Q_0$ without loops, we define a map $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by $x \mapsto x - (x,\alpha_i)\alpha_i$. This map is a reflection with respect to the hyperplane $(\alpha,\cdot) = 0$. It maps $\sum_{i\in Q_0} x_i\alpha_i$ to $\sum_{i\in Q_0} x'_i\alpha_i$, where $x_j = x'_j$ if $j \neq i$ and $x'_i = \sum_j n_{ij}x_j - x_i$. The subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the reflections s_i is denoted by W (or $W(Q)$) and is called the Weyl group of Q.

Now let us define the real and imaginary roots of Q. A real root is a $W(Q)$-conjugate of some α_i, where i has no loops. Note that $(\alpha,\alpha) = 2$ for any real root α. To define imaginary roots without referring to the corresponding Kac-Moody algebra is more complicated. We define the *support* $\mathrm{Supp}\,\alpha$ of an element $\alpha = \sum_i x_i\alpha_i \in \mathfrak{h}^*$ as the set of all i such that $x_i \neq 0$. So we can speak about connected and disconnected (as subgraphs of Q w/o orientation) supports. By an imaginary root we mean a nonzero element $\alpha \in \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\alpha_i)_{i\in Q_0}$ with connected support such that $(\alpha,\alpha) \leqslant 0$.

Lemma 2.1. *Let* α *be an imaginary root. Then* $W(Q)\alpha \subset \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\alpha_i)_{i\in Q_0}$.

The proof is a part of the homework.

2.2. **Kac theorem.** Here's one of the main result about indecomposable representations of quivers.

Theorem 2.2. *The following is true.*

(a) *There is an indecomposable representation with dimension vector* $v \in \mathbb{C}^{Q_0}$ *if and only if* $\sum_i v_i\alpha_i$ *is a root.*

(b) *If v is a real root, then there is a unique indecomposable representation with dimension vector v.*
(c) *In general, if v is a root, then $p_v = 1 - \frac{1}{2}(v,v)$.*

Example 2.3. If Q is of type A_2, then there are three roots: $(1,1),(0,1),(1,0)$, and Kac's theorem clearly holds.

Example 2.4. Let Q be the Jordan quiver (type $\tilde{A}_0$). Then the positive root system coincides with $\mathbb{Z}_{>0}$. We have $(n,n)=0$ for any n. The Kac theorem predicts that the number of parameters describing the orbits of indecomposable representations equals $1-0=1$. We have seen that this is indeed the case.

We are not going to prove all statements of Kac's theorem. We will only check that the dimension of any indecomposable representation is a root, we will prove part (b), and we will check (c) assuming (a) holds and v is primitive (i.e. $\mathrm{GCD}(v_i)_{i\in Q_0}=1$).

We also note that the answer in Kac's theorem does not depend on an orientation of Q. As the homework shows, in many examples some orientations are easier than the others.

3. Deformed preprojective algebras

3.1. **Definition.** We fix an element $\lambda \in \mathbb{C}^{Q_0}$ and define an algebra $\Pi^\lambda(Q)$ (due to Crawley-Boevey and Holland) depending on λ. This algebra is known as the deformed preprojective algebra.

First, let us define the double quiver $\overline{Q}$. We have $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \sqcup Q_1^{op}$, where Q_1^{op} is identified with Q_1 as a set (we denote the corresponding bijection by $a \mapsto a^*$) with $t(a^*) = h(a), h(a^*) = t(a)$ (in other words, for every arrow in Q, we add the opposite arrow).

We set

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/(\sum_{a\in Q_1}[a,a^*] - \sum_{i\in Q_0}\lambda_i\epsilon_i).$$

This relation can be expanded as the Q_0-tuple of relations $\sum_{a,h(a)=i} aa^* - \sum_{a,t(a)=i} a^*a = \lambda_i\epsilon_i$.

Example 3.1. Consider the A_2-quiver. Then $\mathbb{C}\overline{Q}$ is generated by the elements $\epsilon_1,\epsilon_2,a,b$ with relations $\epsilon_2 a = a\epsilon_1 = a, \epsilon_1 b = b\epsilon_2 = b$. In $\Pi^\lambda(Q)$, we have two more relations $ba = \lambda_1\epsilon_1, ab = \lambda_2\epsilon_2$. Here the algebra $\Pi^\lambda(Q)$ is finite dimensional but its dimension depends on λ_1,λ_2.

Example 3.2. Consider the Jordan quiver. Here $\Pi^\lambda(Q) = \mathbb{C}\langle a,a^*\rangle/([a,a^*]=\lambda)$, the first Weyl algebra (for $\lambda \neq 0$) and the polynomial algebra for $\lambda = 0$.

An important property of $\Pi^\lambda(Q)$ is that it is independent of the orientation of Q (up to a natural isomorphism). Namely, suppose that we have changed an orientation of one arrow, say a, getting a new quiver Q'. Then the map $b\mapsto b, b^*\mapsto b, (b\neq a), a\mapsto a^*, a^*\mapsto -a$ defines an isomorphism $\mathbb{C}\overline{Q}\cong \mathbb{C}\overline{Q}'$ that induces an isomorphism $\Pi^\lambda(Q)\cong\Pi^\lambda(Q')$.

3.2. **Moment maps.** The subvariety $\mathrm{Rep}(\Pi^\lambda(Q),v)\subset \mathrm{Rep}(\overline{Q},v)$ is given by the equations $\sum_{a,h(a)=i} x_a x_{a^*} - \sum_{a,t(a)=i} x_{a^*}x_a = \lambda_i \,\mathrm{id}_{V_i}, i\in Q_0$. Let us denote the left hand side by $\mu_i(x_a,x_{a^*})$.

Set $\mu := (\mu_i)_{i\in Q_0} : \mathrm{Rep}(\overline{Q},v)\to \mathfrak{g}_v$. It turns out that $\mu : \mathrm{Rep}(\overline{Q},v)\to\mathfrak{g}_v$ is the so called moment map for the action of $G_v = \prod_{i\in Q_0}\mathrm{GL}(V_i)$. Let us explain what this means.

First of all, let us notice that $\mathrm{Rep}(\overline{Q},v)$ is a symplectic vector space. Indeed, for any finite dimensional vector spaces U,U', we can identify $\mathrm{Hom}(U,U')^*$ with $\mathrm{Hom}(U',U)$ via the trace form. So $\mathrm{Rep}(\overline{Q}, v) = \mathrm{Rep}(Q, v) \oplus \mathrm{Rep}(Q^{op}, v) = \mathrm{Rep}(Q, v) \oplus \mathrm{Rep}(Q, v)^*$. It follows that $\mathrm{Rep}(\overline{Q}, v)$ comes equipped with a natural non-degenerate skew-symmetric form. Using our identifications, it can be written as follows $\omega((x_a, x_{a^*}), (y_a, y_{a^*})) = \sum_{a \in Q_1} \mathrm{tr}(x_a y_{a^*} - y_a x_{a^*})$. Clearly, this form is G_v-invariant.

Now let U be a symplectic vector space with form ω together with a homomorphism $G \to \mathrm{Sp}(U)$ of algebraic groups. By a moment map for this action we mean a G-equivariant map $\mu : U \to \mathfrak{g}^*$ with the property that

$$\langle d_v \mu(u), \xi \rangle = \omega_v(\xi v, u) \tag{3.1}$$

(this condition prescribes $d\mu$ completely). There is a simple formula for μ: $\langle \mu(v), \xi \rangle = \frac{1}{2}\omega(\xi v, v)$, this formula immediately implies both properties.

Lemma 3.3. *The map $\mu = (\mu_i)_{i \in Q_0}$ is a moment map for the G_v-action on the symplectic vector space $\mathrm{Rep}(\overline{Q}, v)$ under the identification of $\mathfrak{g}_v$ with $\mathfrak{g}_v^*$ by means of the trace form.*

Proof. What we need to prove is that

$$\sum_{i \in Q_0} \mathrm{tr} \left(\sum_{h(a)=i} x_a x_{a^*} y_i - \sum_{t(a)=i} x_{a^*} x_a y_i \right) =$$
$$\frac{1}{2}\omega((y_{h(a)} x_a - x_a y_{t(a)}, y_{t(a)} x_{a^*} - x_{a^*} y_{h(a)}), (x_a, x_{a^*}), (x_a, x_{a^*}))$$

By definition, the right hand side is

$$\frac{1}{2} \sum_{a \in Q_1} \left(\mathrm{tr}([y_{h(a)} x_a - x_a y_{t(a)}) x_{a^*}] - \mathrm{tr}([y_{t(a)} x_{a^*} - x_{a^*} y_{h(a)}] x_a) \right).$$

Rearranging the summation and using the cyclicity of trace, we see that the previous expression coincides with the l.h.s. □

Let us explain why moment maps are important. Let $H_\xi = \langle \mu, \xi \rangle$, this is a function on U. To this function we can assign the skew-gradient $v(H_\xi)$ with respect to ω. Condition (3.1) means precisely that $v(H_\xi)$ coincides with the vector field $u \mapsto \xi u$ on U.