LECTURE 15: REPRESENTATIONS OF $U_q(\mathfrak{g})$ AT ROOTS OF 1

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INTRODUCTION

In Lecture 13 we have studied the representation theory of $U_q(\mathfrak{g})$ when q is not a root of 1. We have seen that the representation theory basically looks like the representation theory of $\mathfrak{g}$ (over $\mathbb{C}$).

In this lecture we are going to study a more complicated case: when q is a root of 1 (we still need to exclude some small roots of 1 to make the algebra $U_q(\mathfrak{g})$ defined). When we deal with the usual definition of $U_q(\mathfrak{g})$ we see features of the algebra $U(\mathfrak{g}_{\mathbb{F}})$, where $\mathbb{F}$ is a field of positive characteristic. In particular, $U_q(\mathfrak{g})$ has an analog of the p-center and is finite over its center.

Or we can modify our definition of $U_q(\mathfrak{g})$ including divided powers, this is the case we are going to mostly care about. The corresponding algebra has many features of the semisimple algebraic groups over field of positive characteristic (such as the Frobenius homomorphism). In fact, it is the connection with quantum groups that allowed to compute the characters of irreducible representations of $G_{\mathbb{F}}$ when $\operatorname{char} \mathbb{F} \gg 0$.

For the most part of this lecture we consider the case of $U_q(\mathfrak{sl}_2)$ that can be treated by hand. In the second part we consider a far more complicated case of $U_q(\mathfrak{sl}_n)$. We mention the connection between the representation theory of $U_q(\mathfrak{sl}_n)$ and that of the affine Lie algebra $\hat{\mathfrak{sl}}_n$ discovered by Kazhdan-Lusztig. This is the main tool to study the representation theory of $U_q(\mathfrak{sl}_n)$ (and of other $U_q(\mathfrak{g})$, there we need the affine Lie algebra $\hat{\mathfrak{g}}$).

1. CASE OF $U_q(\mathfrak{sl}_2)$

Recall that $U_q(\mathfrak{sl}_2)$ is defined by generators $E, K^{\pm 1}, F$ and relations

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

These relations give us some freedom of choosing the base ring. In this lecture, q will be viewed as an element of $R := \mathbb{C}[q^{\pm 1}][(q - q^{-1})^{-1}]$, it obviously makes sense to consider $U_q(\mathfrak{sl}_2)$ as an R-algebra. This algebra will be denoted by U_R. For $\epsilon \in \mathbb{C} \setminus \{0, \pm 1\}$, we set $U_\epsilon := \mathbb{C}_\epsilon \otimes_R U_R$. By $\mathbb{K}$ we denote the fraction field $\mathbb{C}(q)$ of R and we set $U_{\mathbb{K}} = \mathbb{K} \otimes_R U_R$.

For the remainder of this section, we write $\mathfrak{g}$ for $\mathfrak{sl}_2$ and G for SL_2 (over $\mathbb{C}$ by default).

1.1. Usual $U_\epsilon(\mathfrak{sl}_2)$.
Let ϵ denote a primitive root of 1 of order d. We set $d_0 = d/2$ if d is even and $d_0 = d$ if d is odd.

Proposition 1.1. *The elements $E^{d_0}, K^{d_0}, F^{d_0} \in U_\epsilon$ are central.*

The proof is a part of the homework.

Let Z_{d_0} denote the subalgebra of U_ϵ generated by $F^{d_0}, K^{d_0}, E^{d_0}$. It is an analog of the p-center of $U(\mathfrak{sl}_2(\mathbb{F}))$.

Proposition 1.2. *Elements $F^{kd_0}K^{\ell d_0}E^{md_0}$, where $k, m \in \mathbb{Z}_{\geqslant 0}, \ell \in \mathbb{Z}$, form a basis in Z_{d_0}. The algebra U_ϵ is a free Z_{d_0}-module with basis $F^{k_1}K^{\ell_1}E^{m_1}$, where $k_1, \ell_1, m_1 \in \{0, \ldots, d_0-1\}$.*

The proof of the first claim is based on the formula for $E^r F^s$ that will be provided below. All other claims are easy.

The representation theory of U_ϵ is very similar to that of $U(\mathfrak{g}_\mathbb{F})$, where $\mathfrak{g} = \mathfrak{sl}_2$, but we are not going to provide details here. One thing that we will need is that the modules $L_0(\kappa\epsilon^s)$ over U_ϵ are irreducible when $s \in \{0, \ldots, d_0-1\}$ (and the central elements $F^{d_0}, K^{d_0} - \kappa^{d_0}, E^{d_0}$ act by 0).

1.2. **Divided power algebra.** Now we are going to define a different algebra to be a denoted by $\dot{U}_R$ and its specializations. The algebra $\dot{U}_R$ is defined as the subalgebra inside $U_\mathbb{K}$ generated $K^{\pm 1}$ by the divided powers $E^{(k)} := E^k/[k]_q!, F^{(k)} := F^k/[k]_q!$. By the very definition, $\mathbb{K} \otimes_R \dot{U}_R := U_\mathbb{K}$. We set $\dot{U}_\epsilon := \mathbb{C}_\epsilon \otimes_R \dot{U}_R$. Note that if ϵ is not a root of 1, the algebra $\dot{U}_\epsilon$ coincides with U_ϵ, while for a root of 1 we get something different. This is because still have elements $E^{(k)}, F^{(k)}$ in $\dot{U}$ but they are no longer polynomials in E, F. Since $[d_0]_\epsilon! = \frac{\epsilon^{d_0} - \epsilon^{-d_0}}{\epsilon - \epsilon^{-1}} = 0$, we get $E^{d_0} = F^{d_0} = 0$ in $\dot{U}_\epsilon$. Below we will only (for simplicity) consider the case when d is odd so that $d = d_0$. Note that $[k]_\epsilon = 0$ if and only if k is divisible by d.

Remark 1.3. This construction is strongly motivated by the representation theory of algebraic groups in characteristic p. There one defines the *hyperalgebra* $\dot{U}_\mathbb{Z}(\mathfrak{g}) \subset U(\mathfrak{g}_\mathbb{Q})$ and its specialization $\dot{U}_\mathbb{F}(\mathfrak{g})$. The point is that the category of finite dimensional $\dot{U}_\mathbb{F}(\mathfrak{g})$-modules coincides with that of the rational representations of $G_\mathbb{F}$.

Lemma 1.4. *The algebra $\dot{U}_\epsilon$ is generated by $E, K^{\pm 1}, F, E^{(d)}, F^{(d)}$.*

Proof. Note that

$$[kd]_q = \frac{q^{kd} - q^{-kd}}{q - q^{-1}} = \frac{q^{kd} - q^{-kd}}{q^d - q^{-d}}\frac{q^d - q^{-d}}{q - q^{-1}} = [k]_{q^d}[d]_q.$$

So

(1.1) $$([kd]_q/[d]_q)\,|_{q=\epsilon} = k.$$

It follows that any $E^{(k)}$ is a polynomial in E and $E^{(d)}$ and the similar statement holds for $F^{(k)}$. $\qquad\square$

Our next goal is to establish the triangular decompositions

$$\dot{U}_R = \dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes \dot{U}_R^+, \quad \dot{U}_\epsilon = \dot{U}_\epsilon^- \otimes \dot{U}_\epsilon^0 \otimes \dot{U}_\epsilon^+$$

(as R-modules/vector spaces). In order to do this we will need some identities in $U_\mathbb{K}$. First, some notation. For $a \in \mathbb{Z}$, we set

$$[K; a] := \frac{Kq^a - K^{-1}q^{-a}}{q - q^{-1}}$$

so that $[E, F] = [K; 0]$. Consider the "binomial coefficient"

$$\binom{K; a}{i} = \left(\prod_{j=0}^{i-1}[K; a - j]\right)/[i]_q!.$$

Lemma 1.5. *We have the following equalities in $U_{\mathbb{K}}$:*

(1.2)
$$E^{(r)}F^{(s)} = \sum_{i=0}^{\min(r,s)} F^{(s-i)}\binom{K;2i-r-s}{i}E^{(r-i)},$$

(1.3)
$$\binom{K;a+1}{i} = q^i\binom{K;a}{i} + q^{i-a-1}K^{-1}\binom{K;a}{i-1},$$

(1.4)
$$\binom{K;a}{i}E^{(k)} = E^{(k)}\binom{K;a+2k}{i},$$

(1.5)
$$\binom{K;a}{i}F^{(k)} = F^{(k)}\binom{K;a-2k}{i}.$$

Proof. (1.2) is proved by the double induction. (1.3), a q-analog of a classical binomial identity, is obtained by a direct calculation. (1.4),(1.5) are straightforward corollaries of $KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F$, respectively. $\qquad\square$

(1.2) and (1.3) imply that all $\binom{K;a}{i} \in \dot{U}_R$. We will denote the specialization of $\binom{K;a}{i}$ to $\dot{U}_\epsilon$ by the same symbol.

Now let us proceed to the triangular decompositions. Set $\dot{U}_R^? := \dot{U}_R \cap U_{\mathbb{K}}^?$, where $? = +,0,-$, and $U_{\mathbb{K}}^+, U_{\mathbb{K}}^0, U_{\mathbb{K}}^-$ are the subalgebras of $U_{\mathbb{K}}$ generated by E, K, F, respectively so that we have $U_{\mathbb{K}} = U_{\mathbb{K}}^- \otimes_{\mathbb{K}} U_{\mathbb{K}}^0 \otimes U_{\mathbb{K}}^+$. We write $\dot{U}_\epsilon^? = \mathbb{C}_\epsilon \otimes_R \dot{U}_R^?$.

Proposition 1.6. *We have the following triangular decompositions.*

(1) $\dot{U}_R = \dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes_R \dot{U}_R^+$. *The algebra $\dot{U}_R^\pm$ is a free R-module with basis $E^{(k)}$ (or $F^{(k)}$), where $k \in \mathbb{Z}_{\geqslant 0}$. The R-module $\dot{U}_R^0$ is spanned by the elements of the form $K^\ell\binom{K;0}{i}$, where $\ell, i \in \mathbb{Z}$.*

(2) $\dot{U}_\epsilon = \dot{U}_\epsilon^- \otimes \dot{U}_\epsilon^0 \otimes \dot{U}_\epsilon^+$. *The space $\dot{U}_\epsilon^\pm$ has basis $E^k(E^{(d)})^\ell$ (or $F^k(F^{(d)})^\ell$) with $k \in \{0,1,\dots,\ell-1\}$ or $\ell \in \mathbb{Z}_{\geqslant 0}$. The space $\dot{U}_\epsilon^0$ has basis $K^m\binom{K;0}{d}^\ell$, where $0 \leqslant d \leqslant 2m-1$.*

Proof. Let us prove (1). The algebra $\dot{U}_R$ is spanned by $K^s E^{(\ell_1)}F^{(m_1)}\dots E^{(\ell_n)}F^{(m_n)}$. Then we use relations of Lemma 1.5 to show that $\dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes_R \dot{U}_R^+ \twoheadrightarrow U_R$. This map is an isomorphism after base change to K. All factors are torsion free over R and so is their product. We deduce that $\dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes_R \dot{U}_R^+ = U_R$. The claims about bases/spanning sets are left as exercises.

Let us prove (2). The triangular decomposition is a straightforward corollary of the claim for $\dot{U}_R$. The claim about the bases in $\dot{U}_\epsilon^\pm$ follows from the corresponding claims for $\dot{U}_R^\pm$ and (1.1). The claim for $\dot{U}_\epsilon^0$ is more subtle and is left as an exercise (note for example that we have $\prod_{i=0}^{d-1}[K;-i] = 0$ in $\dot{U}_\epsilon^0$ hence we can only use $K^m, m = 0,\dots,2d-1$, in generators). $\qquad\square$

Now let us present $\dot{U}_\epsilon$ by generators and relations.

Proposition 1.7. *The algebra $\dot{U}_\epsilon$ is generated by $E, F, K^{\pm 1}, E^{(d)}, F^{(d)}, \binom{K;0}{d}$ with the following relations:*

$$(1.6) \qquad EE^{(d)} = E^{(d)}E, \quad FF^{(d)} = F^{(d)}F, \quad K\binom{K;0}{d} = \binom{K;0}{d}K,$$

$$(1.7) \qquad KEK^{-1} = \epsilon^2 E, \quad KFK^{-1} = \epsilon^{-2}F, \quad KE^{(d)}K^{-1} = E^{(d)}, \quad KF^{(d)}K^{-1} = F^{(d)},$$

$$(1.8) \qquad E\binom{K;0}{d} = \binom{K;2}{d}E,$$

$$(1.9) \qquad \binom{K;0}{d}F = F\binom{K;2}{d},$$

$$(1.10) \qquad \binom{K;0}{d}E^{(d)} = E^{(d)}\binom{K;0}{d} + 2E^{(d)}, \quad \binom{K;0}{d}F^{(d)} = F^{(d)}\binom{K;0}{d} - 2F^{(d)}.$$

$$(1.11) \qquad [E,F] = [K;0], \quad [E, F^{(d)}] = F^{(d-1)}[K; 1-d], \quad [E^{(d)}, F] = [K; 1-d]E^{(d-1)},$$

$$(1.12) \qquad [E^{(d)}, F^{(d)}] = \binom{K;0}{d} + \sum_{i=1}^{d-1} F^{(d-i)}\binom{K; 2i-2d}{i}E^{(d-i)}.$$

In (1.8,1.9) one recovers $\binom{K;2}{d}$ from $\binom{K;0}{d}$ and polynomials in K using (1.3). In the summation in the right hand side of (1.12) and in the right hand side of (1.11) we have an expression of F, K, E.

Proof. Let $\tilde{U}_\epsilon$ be the algebra generated by the generators and relations above. All the relations above hold in $\dot{U}_\epsilon$ and so we get $\tilde{U}_\epsilon \twoheadrightarrow \dot{U}_\epsilon$. Using the relations for $\tilde{U}_\epsilon$ we can see that elements of the form $F^{k_1}(F^{(d)})^{k_2}K^{\ell_1}\binom{K;0}{d}^{\ell_2}E^{m_1}(E^{(d)})^{m_2}$ with $k_1, m_1 \in \{0, \ldots, d-1\}, \ell_1 \in \{0, \ldots, 2d-1\}, k_2, \ell_2, m_2 \in \mathbb{Z}_{\geq 0}$, span $\tilde{U}_\epsilon$. Hence $\tilde{U}_\epsilon \xrightarrow{\sim} \dot{U}_\epsilon$. $\qquad\square$

One can check that the subalgebra $\dot{U}_R \subset U_{\mathbb{K}}$ is a Hopf subalgebra. It follows that $\dot{U}_\epsilon$ is a Hopf algebra.

Now let us investigate the Hopf algebra structures on $\dot{U}_R, \dot{U}_\epsilon$.

Lemma 1.8. *$\dot{U}_R \subset U_{\mathbb{K}}$ is a Hopf subalgebra. So $\dot{U}_\epsilon$ becomes a Hopf algebra.*

Proof. This boils down to show that $\Delta(\dot{U}_R) \subset \dot{U}_R \otimes_R \dot{U}_R$ and $S(\dot{U}_R) \subset \dot{U}_R$. We will do the first check. It will follow if we check that $\Delta(E^{(k)}), \Delta(F^{(k)}) \subset \dot{U}_R \otimes_R \dot{U}_R$. We have

$$\Delta(E^{(k)}) = ([k]_q!)^{-1}(E \otimes 1 + K \otimes E)^k = \sum_{i=0}^{k} q^{i(k-i)}E^{(k-i)}K^i \otimes E^{(i)},$$

$$\Delta(F^{(k)}) = ([k]_q!)^{-1}\sum_{i=0}^{k} q^{i(k-i)}F^{(k-i)} \otimes F^{(i)}K^{-i}.$$

Both right hand sides are in $\dot{U}_R \otimes_R \dot{U}_R$. $\qquad\square$

1.3. **Small quantum group and quantum Frobenius.** By the small quantum group u_ϵ we mean the subalgebra of $\dot{U}_\epsilon$ generated by $E, K^{\pm 1}, F$. Another way to define it is as the image of the natural homomorphism $U_\epsilon \to \dot{U}_\epsilon$. Note that u_ϵ is a Hopf quotient of U_ϵ and a Hopf subalgebra of $\dot{U}_\epsilon$.

Proposition 1.9. *The algebra u_ϵ is the quotient of U_ϵ by $(E^d, F^d, K^{2d}-1)$. It has basis $F^k K^\ell E^m$, where $k, m \in \{0, \ldots, d-1\}$ and $\ell \in \{0, \ldots, 2d-1\}$.*

Now we want to describe the the quotient of $\dot{U}_\epsilon$ by $(E, F, K - 1)$.

Proposition 1.10. *We have $\dot{U}_\epsilon/(E, F, K - 1) = U(\mathfrak{sl}_2)$ with $E^{(d)} \mapsto e, F^{(d)} \mapsto f$ and $\binom{K;0}{d} \mapsto h$. This is an isomorphism of Hopf algebras.*

Proof. By Proposition 1.7, $\dot{U}_\epsilon/(E, F, K-1)$ is generated by the elements $E^{(d)}, F^{(d)}, \binom{K;0}{d}$ that satisfy the defining relations of $U(\mathfrak{sl}_2)$. This gives the required isomorphism. It preserves the Hopf algebra structure because it intertwines $\Delta(E^{(d)})$ with $\Delta(e)$ and $\Delta(F^{(d)})$ with $\Delta(f)$. The latter is a consequence of the computation in the proof of Lemma 1.8. $\square$

The epimorphism $\dot{U}_\epsilon \twoheadrightarrow U(\mathfrak{sl}_2)$ is called the *quantum Frobenius* epimorphism and is denoted by Fr.

1.4. **Classification of the irreducibles and characters.** The classification of finite dimensional irreducible $\dot{U}_\epsilon$-modules works similarly to to that of $\mathrm{SL}_2(\mathbb{F})$). Besides, we have an analog of the Steinberg decomposition.

Theorem 1.11. *There is a bijection between $\mathrm{Irr}_{f.d.}(\dot{U}_\epsilon)$ and the set $\{\pm 1\} \times \mathbb{Z}_{\geqslant 0}$. An element (κ, n) in the latter set goes to the unique finite dimensional irreducible module $L(\kappa, n)$ that has a vector $v_{\kappa,n}$ with (here $n = dm + r$, where $0 \leqslant r < d$,)*

$$(1.13) \qquad K v_{\kappa,n} = \kappa\epsilon^n v_{\kappa,n}, \quad \binom{K;0}{d} v_{\kappa,n} = m v_{\kappa,n}, \quad E^{(i)} v_{\kappa,n} = 0, i > 0.$$

then $L(\kappa, n) = L(\kappa, r) \otimes \mathrm{Fr}^* L(m)$, where $E^{(d)}, F^{(d)}$ act by 0 on $L(\kappa, r)$ and $L(m)$ stands for the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module with highest weight m.

To prove this theorem is a part of the homework.

Moreover, we note that we still have the universal R-matrix for $\dot{U}_\epsilon$ (but not for U_ϵ!). Indeed, Θ lies in a completion of $\dot{U}_R$ (and becomes a finite sum in $\dot{U}_\epsilon$). Moreover, one can make sense for Ψ for $\dot{U}_\epsilon$. The resulting R-matrix enjoys the same properties as for $q \neq \sqrt[\bullet]{1}$.

2. CASE OF $U_q(\mathfrak{sl}_n)$

In this section we deal with $\mathfrak{g} = \mathfrak{sl}_n$. The case of a more general semisimple Lie algebra is treated similarly but we want to keep the exposition simpler.

We can define the subalgebra $\dot{U}_R(\mathfrak{sl}_n) \subset U_\mathbb{K}(\mathfrak{sl}_n)$ similarly to the above. We then consider the specialization $\dot{U}_\epsilon(\mathfrak{sl}_n)$, where ϵ is a primitive dth root of 1 (we still assume that d is odd). This specialization admits a triangular decomposition $\dot{U}_\epsilon(\mathfrak{sl}_n) = \dot{U}_\epsilon^- \otimes \dot{U}_\epsilon^0 \otimes \dot{U}_\epsilon^+$, where $\dot{U}_\epsilon^-$ is generated by the elements $F_i^{(k)}$, $\dot{U}_\epsilon^0$ is generated by the elements $K_i^{\pm 1}$ and $\binom{\{K_i;a\}}{j\}}$, $\dot{U}_\epsilon^+$ is generated by the elements $E_i^{(k)}$.

The finite dimensional irreducible representations are still parameterized by $\{\pm 1\}^{n-1} \times P^+$, where we write P^+ for the monoid of dominant weights. We will be interested in the representations parameterized by P^+. Every such module V admits a weight decomposition $V = \bigoplus_\mu V_\mu$, where V_μ is determined as follows. Let $\mu = (\mu_1, \ldots, \mu_n)$. Then V_μ consists of all elements $v \in V$ such that $K_i v = \epsilon^{r_i} v$ and $\binom{K;0}{d} v = d m_i v$. Here we write $\mu_i = dm_i + r_i$ for the division with remainder. For $\lambda \in P^+$, we write $L(\lambda)$ for the corresponding irreducible representation.

The module $L(\lambda)$ can be realized as the unique irreducible quotient of the Weyl module $W(\lambda)$. The latter is defined by a single vector v_λ and the following relations: $v \in W(\lambda)_\lambda$,

$E_i^{(k)} v_\lambda = 0$ for all $k > 0$ and $F_i^{(k)} v_\lambda = 0$ for $k > \mu_i$. Note that those are precisely the relations defining the irreducible module with highest weight λ over $U(\mathfrak{sl}_n)$.

The module $W(\lambda)$ enjoys the same universal property as the Weyl module for the representations of $\mathrm{SL}_n(\mathbb{F})$:

$$\mathrm{Hom}_{\dot{U}_\epsilon}(W(\lambda), V) = \{v \in V_\lambda | E_i^{(k)} v = 0, \forall i = 1, \ldots, n, \forall k > 0\}.$$

As usual, an important problem is to compute the characters of the modules $L(\lambda)$. This boils down to determining the multiplicities of $L(\mu)$'s in $W(\lambda)$'s. The solution follows from the work of Kazhdan and Lusztig, [KL] (for an arbitrary finite dimensional semisimple Lie algebra $\mathfrak{g}$). They established an equivalence of $\dot{U}_\epsilon$-mod$_{f.d.}$ and a certain category of representations of the affine Lie algebra $\hat{\mathfrak{sl}}_n$ (that was defined in Homework 5). The multiplicities in the latter are given by values at 1 of suitable Kazhdan-Lusztig polynomials for the affine symmetric group $\hat{S}_n$.

2.1. **Connection to representation theory of $\hat{\mathfrak{sl}}_n$.** As usual, an important problem is to compute the characters of the modules $L(\lambda)$. This boils down to determining the multiplicities of $L(\mu)$'s in $W(\lambda)$'s. The solution follows from the work of Kazhdan and Lusztig, [KL] (for an arbitrary finite dimensional semisimple Lie algebra $\mathfrak{g}$). They established an equivalence of $\dot{U}_\epsilon$-mod$_{f.d.}^1$ (the category of modules, where K acts with eigenvalues that are powers of ϵ) and a certain category of representations of the affine Lie algebra $\hat{\mathfrak{sl}}_n$ (that was defined in Homework 5). The multiplicities in the latter are given by values at 1 of suitable Kazhdan-Lusztig polynomials for the affine symmetric group $\hat{S}_n$.

We are going to finish by describing the Kazhdan-Lusztig category $\mathcal{C}$ on the side for $\mathfrak{g} = \hat{\mathfrak{sl}}_n$. Pick $\kappa \notin \mathbb{Q}_{>0}$ such that $\exp(\pi\sqrt{-1}\kappa^{-1}) = \epsilon$. By definition, $\mathcal{C}$ consists of the modules with the following properties:

- $\mathfrak{m}_+ := t\mathfrak{g}[t]$ acts locally nilpotently.
- $\mathfrak{g} \subset \hat{\mathfrak{g}}$ acts locally finitely (any vector lies in a finite dimensional $\mathfrak{g}$-submodule).
- $c \in \hat{\mathfrak{g}}$ acts by the scalar $\kappa - m$.

Let us produce an example of a module in $\mathcal{C}$, the Weyl module (a.k.a. a parabolic Verma module) $\Delta(\lambda)$. Let $V(\lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight λ. We turn it into a $\mathfrak{g}[t]$-module by making $t\mathfrak{g}[t]$ act by 0. We make c to act by $\kappa - m$. Then set $\Delta(\lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}c)} V(\lambda)$.

The following is the main result of [KL] (for an arbitrary semisimple Lie algebra $\mathfrak{g}$).

Theorem 2.1. *There is a category equivalence between $U_q(\mathfrak{g})$-mod$_{f.d.}^1$ and the category $\mathcal{C}$ that maps $W(\lambda)$ to $\Delta(\lambda)$.*

The main idea of the proof is to recover the tensor product on the level of $\mathcal{C}$. This is done using a construction called *conformal blocks* that comes from Math Physics.

<div align="center">References</div>

[KL] D. Kazhdan, G. Lusztig. *Tensor structures arising from affine Lie algebras*, I,II: J. Amer. Math. Soc., 6 (1993), no. 4, 905-947, 949-1011, III,IV: J. Amer. Math. Soc. 7 (1994), no. 2, 335-381, 383-453.