LECTURE 14: LINK INVARIANTS FROM QUANTUM GROUPS

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Introduction

In this lecture we explain how to construct invariants of links from representations of quantum groups. We use the representation $V = L(q)$ of $U_q(\mathfrak{sl}_2)$ to produce the invariant known as the Jones polynomial.

We start in Section 1 by recalling the basic notions of knot theory and introducing the Jones polynomial. The definition can be used to show its uniqueness but not existence.

One way to prove the existence of the Jones polynomial is to relate links to braids. Any link can be obtained as a braid closure. A link invariant then corresponds to a *Markov trace* on the braid groups, a collection of maps $B_n \to X$ (where X is some set) satisfying certain compatibility relations. We produce such a trace from the B_n-action on $V^{\otimes n}$.

In the last section we explain another way to produce link invariants from representations of quantum groups due to Reshetikhin and Turaev. It is better computationally, one can compute the invariant directly from the diagram. More generally, a construction produces a homomorphism of suitable quantum group modules from a tangle.

1. Background from knot theory

1.1. **Links and their diagrams.** By a *link* we mean a continuous embedding of $\mathbb{S}^1 \sqcup \ldots \sqcup \mathbb{S}^1$ (the disjoint union of k circles) into $\mathbb{R}^3$. A link with a single component is called a knot. We view links up to isotopy (a continuous family of diffeomorphisms of $\mathbb{R}^3$). We can also consider oriented knots and links.

Usually, knots and links are presented by their two dimensional diagrams by picking a suitable projection $\mathbb{R}^3 \to \mathbb{R}^2$. Namely, we consider projections that have simple transverse intersections, i.e. we do not allow tangent strands or three strands intersecting in a single point. See examples in Picture 1.1.

We can speak about isotopic diagrams – we use continuous families of diffeomorphisms of $\mathbb{R}^2$. But isotopic links may have a non-isotopic diagrams. One can consider so called *Reidemeister moves* (see Picture 1.2), they take a piece of a diagram and transform it in such a way that change an isotopy class of a diagram but not of a link.

Theorem 1.1. *Two diagrams correspond to isotopic (oriented) links if and only if they can be obtained from one another by diagram isotopies and (oriented, just put various orientations on the fragments) Reidemeister moves.*

There is no algorithm however to test whether two diagrams can be obtained from one another as described in the theorem. So one tries to produce invariants of (oriented) diagrams that are preserved by diagram isotopies and Reidemeister moves and that are algorithmically computable.

1.2. **Jones polynomial.** Let us take a small circle in a diagram that contains precisely one intersection of strands. Then the diagram inside the circle looks like one of two fragments L_+ or L_-, see Picture 1.3, that are not isotopic (inside the circle). Another fragment we can have inside the circle is L_0. Now consider the ambient links that are the same outside the circle and are equal to L_+, L_0, L_- inside it. Abusing the notation we still denoted these links by L_+, L_-, L_0.

Theorem 1.2. *There is a unique oriented link invariant $L \mapsto P(L) \in \mathbb{Z}[q^{\pm 1}]$ such that $q^{-2}P(L_+) - q^2 P(L_-) = (q^{-1} - q)P(L_0)$ (skein relation) whose value on the trivial link with n components (unlink) is $(q + q^{-1})^{n-1}$.*

Theorem implies that $P(L)|_{q=1} = 2^{k-1}$, where k is the number of components.

It is possible to compute this invariant algorithmically. Namely, pick a point on a diagram and move this point according to the orientation. When we reach a crossing we put the strand we are on on top if it was on the bottom. If the diagram has changed, we write the skein relation expressing the Jones polynomial of the previous diagram as the sum of two. When we return to the starting point we will get the expression for the original polynomial in terms of a bunch of summands with one less crossing and a summand, where the link component we are on became untangled (meaning that it gives a trivial embedding $\mathbb{S}^1 \hookrightarrow \mathbb{R}^3$ that is not linked to other components).

Example 1.3. We compute the Jones polynomial of the Hopf link oriented as in Picture 1.1 (two different orientations may – and will – give different Jones polynomials). Let us consider the upper crossing point, see Picture 1.4. Then our initial Hopf link gives L_- so we will write $\tilde{L}_-$ for that link. Switching the crossing to L_+, we'll get the link $\tilde{L}_+$ that is two unlinked circles. Switching the crossing to L_0, we'll get $\tilde{L}_0$ that is the unknot. So $P(\tilde{L}_+) = q + q^{-1}$ and $P(\tilde{L}_0) = 1$. From the skein relation, we find

$$q^{-2}P(\tilde{L}_+) - q^2 P(\tilde{L}_-) = (q^{-1} - q)P(\tilde{L}_0) \Rightarrow P(\tilde{L}_-) = q^{-4}(q + q^{-1}) - q^{-2}(q^{-1} - q) = q^{-5} + q^{-1}.$$

Example 1.4. For the trefoil K in Picture 1.1 we have $P(K) = q^2 + q^6 - q^8$, see Picture 1.5 for some explanation.

2. Jones polynomial as Markov trace

2.1. **Braids, geometrically.** Recall the braid group B_n introduced in the previous lecture. It admits a geometric presentation similar (and closely related) to links. We will write B_n^g for this realization. As a set B_n^g consists of the configurations of n strands in $\mathbb{R}^2 \times [0, 1]$ connecting points $(i, 0, 0)$ to points $(j, 0, 1)$ (one-to-one), where $i, j = 1, \ldots, n$, in some order in such a way that

(a) each strand projects isomorphically to $[0, 1]$
(b) and the strands do not intersect.

We identify two braids that are obtained by an isotopy (fixing the $2n$ points and preserving the conditions above). We can present braids by braid diagrams, see Picture 2.1.

Proposition 2.1. *Two braid diagrams give isotopic braids if one is obtained from the other by a sequence of diagram isotopies and Reidemeister moves (R2) and (R3) (condition (a) prohibits the situation in (R1)). Let B_n^g denote the set of all these geometric braids.*

The set B_n^g admits an associative product (concatenation, Picture 2.2). This product has a unit given by the trivial braid (straight strands connecting $(i, 0, 0)$ to $(i, 0, 1)$ for each i).

As a monoid B_n^g is generated by the braids T_i, T_i^{-1} presented on Picture 2.3. That these elements generate B_n^g should be clear from Picture 2.4 (just perturb the diagram so that the projections of all crossings to $[0,1]$ are distinct). (R2) precisely says that T_i and T_i^{-1} are inverse to one another, so our notation is justified. In particular, B_n^g is a group rather than just a monoid. Note that $T_iT_j = T_jT_i$ when $|i-j|>1$ (via a diagram isotopy). Also $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, this is precisely (R3). So we get a group epimorphism $B_n \twoheadrightarrow B_n^g$. The following result is a consequence of Proposition 2.1.

Theorem 2.2. *The epimorphism $B_n \twoheadrightarrow B_n^g$ is an isomorphism.*

2.2. **Braids vs links.** Given a braid b, we orient it from right to left. Then we can take the so called *braid closure*, see Picture 2.5, and get an oriented link. The following result is due to Alexander.

Theorem 2.3. *Any oriented link is the closure of some braid.*

Now let us figure out when two braids $b \in B_n, b' \in B_{n'}$ give the same link. Note that $\overline{ab} = \overline{ba}$, Pic 2.6. Now let us take $b \in B_n$. We can embed B_n into B_{n+1} (just put a strand from $(n+1,0,0)$ to $(n+1,0,0)$ that is below all other strands). Pick $b \in B_n$. We can view b as an element of B_{n+1} and form the product $bT_n^{\pm 1} \in B_{n+1}$. Then $\overline{bT_n^{\pm 1}} = \bar{b}$.

The following important result is due to Markov.

Theorem 2.4. *Braids $b_1 \in B_{n_1}, b_2 \in B_{n_2}$ have the same closure if and only if b_1 can be obtained from b_2 by a sequence of Markov moves*

(M1) *$ab \leftrightarrow ba$, for a, b in same B_n.*
(M2) *$b \leftrightarrow bT_n^{\pm 1}$, for $b \in B_n \hookrightarrow B_{n+1}$.*

By a *Markov trace*, we mean a collection of maps $\varphi_n : B_n \to \mathbb{C}$ (or some other target) that do not change under the Markov moves. By Theorem 2.4, this is the same thing as an oriented link invariant. The reason why we call it a trace is that $\varphi_n(ab) = \varphi_n(ba)$ is satisfied as soon as $\varphi_n(b) = \operatorname{tr}(\Phi_n(b))$ for some representation Φ_n of B_n.

2.3. **Markov trace from $L(q)$.** Let V be the U-module $L(q)$, where we write $U := U_q(\mathfrak{sl}_2)$. We have a homomorphism $B_n \to \mathbb{Z}$ called the degree (and denoted by deg). It is defined on the generators $\deg(T_i) = 1$ (and extends to B_n because all relations preserve the degrees). Now recall from the previous lecture that B_n acts on $V^{\otimes n}$ by U-linear automorphisms: T_i maps to $\tau_{i,i+1} = \mathrm{id}^{\otimes(i-1)} \otimes (R_{V\otimes V} \circ \sigma) \otimes \mathrm{id}^{\otimes n-i-1}$. Denote this representation by Φ_n'. The action of B_n commutes with the action of K that is given by an iterated Δ of K, i.e., by $K^{\otimes n}$. The trace of Φ_n' "almost" give a Markov trace but not quite.

Theorem 2.5. *The maps φ_n given by $\varphi_n(b) = q^{2\deg(b)} \operatorname{tr}(K^{\otimes n}\Phi_n'(b))$ form a Markov trace. Moreover, $\varphi_n(b) = (q+q^{-1})P(\bar{b})$.*

The proof of this theorem (in a more general setting, where we replace $U_q(\mathfrak{sl}_2)$ by $U_q(\mathfrak{sl}_n)$ is a part of the homework).

Example 2.6. Let us compute $\varphi_n(b)$ for $b = 1 \in B_n$. We get $\varphi_n(b) = q^0 \operatorname{tr}(K^{\otimes n}) = \operatorname{tr}(K)^n = (q+q^{-1})^n$.

Now let us compute $\varphi_2(T_1^2)$. We have $\Phi_2'(T_1^2) = 1 + (q^{-1}-q)\Phi_2(T_1)$. So $\varphi_2(T_1^2) = q^4 \operatorname{tr}(K^{\otimes}2) + q^4(q^{-1}-q)\operatorname{tr}(K^{\otimes 2}T_1)$. But we know that φ_n form the Markov trace, so $\varphi_2(T_1) = q^2 \operatorname{tr}(K^{\otimes 2}T_1) = \varphi_1(1) = (q+q^{-1})$. So $\varphi_2(T_1^2) = q^4(q+q^{-1})^2 + q^2(q^{-1}-q)(q+q^{-1}) = (q+q^{-1})(q^5+q)$. Note that the closure of T_1^2 is a Hopf link.

The theorem above proves the existence of the Jones polynomial but is not very useful for computations. In the next section we will consider another construction of the Jones polynomial, which also proves the existence and is better for computations.

3. Tangles and representations of quantum groups

3.1. **Tangles.** Tangles generalize both braids and links. A tangle is the following configuration: it consists of oriented links in $\mathbb{R}^2 \times [0,1]$ and oriented strands that connect some n fixed points on $\mathbb{R}^2 \times \{0\}$ and m fixed points on $\mathbb{R}^2 \times \{1\}$ (we can connect two points on $\mathbb{R}^2 \times \{0\}$ or two points on $\mathbb{R}^2 \times \{1\}$ with an oriented arc), points are connected one-to-one, in particular, $n+m$ has to be even. We consider tangles up to isotopy that fixes the $n+m$ points. We get the set $T(n,m)$ of isotopy classes. Note that $T(0,0)$ consists precisely of the oriented links.

By a *signed set* we mean a set together with a map to $\{\pm\}$. A tangle gives structures of signed sets on $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$: sinks on $\mathbb{R}^2\times\{0\}$ and sources on $\mathbb{R}^2\times\{1\}$ are sent to a $+$, all other points are sent to a $-$. So, for two signed sets, M,N with $|M|=m, |N=n|$, we can define the subset $T(N,M)\subset T(n,m)$ corresponding to given signed sets.

We can still represent tangles by tangle diagrams, see Picture 3.1. Two tangles T_1,T_2 are isotopic if and only if the diagram of T_2 is obtained from that of T_1 by a sequence of diagram isotopies and the Reidemeister moves (R1),(R2),(R3).

We can compose tangles getting a partial composition map $T(K,N)\times T(N,M)\to T(K,M)$ similarly to the braids. Generating tangles are the crossings $X_+,X_-\in T(2,2)$, Picture 3.2, and also caps and cups in $T(2,0)$ and $T(0,2)$ (usually tangles are drawn vertically, hence the names), each with 2 possible orientations. Note that all other crossings are obtained as compositions of $X_\pm$ with cups and caps, see Picture 3.3 (we can rotate the crossing using cups and caps). Now the argument to show that $X_\pm$, cups and caps are generators is the same as for the braids.

We also have the tensor product $T(n_1,m_1)\times T(n_2,m_2)\to T(n_1+n_2,m_1+m_2)$, by definition, the diagram of $T_1\otimes T_2$ is obtained by putting the diagram of T_2 above the diagram of T_1, see Picture 3.4.

3.2. **Functor.** Let $V=L(q)$. We assign V,V^* to the $n+m$ points: V goes to the point labeled by a $+$ and V^* to a point labeled by a $-$. To a signed set M we assign the module to be denoted by $V^{\otimes M}$, which is the tensor product of modules assigned to points in M.

Our goal is, for $T\in T(N,M)$, construct a U-linear homomorphism $\varphi_T: V^{\otimes M}\to V^{\otimes N}$ in such a way that $\varphi_{T_1\circ T_2}=\varphi_{T_1}\circ\varphi_{T_2}$ and $\varphi_{T_1\otimes T_2}=\varphi_{T_1}\otimes\varphi_{T_2}$.

This is done as follows: we need to define φ_T for generating tangles, extend it to arbitrary tangles so that diagrams corresponding to the same tangle give the same homomorphisms. In other words, we need to check that the homomorphism is preserved under a diagram isotopy and respects the three Reidemeister moves. We are not going to discuss this check, it requires a much more careful examination of how tangle isotopies work.

The generating tangles are the cups in $T(0,2)$, caps in $T(2,0)$ and the crossings in $T(2,2)$ (lines should clearly give the identity isomorphism). The homomorphism corresponding to X_+ (and to its rotations) is $q^2\tau_{?\otimes ?}$, while the homomorphism corresponding to X_- is $q^{-2}\tau_{?\otimes ?}^{-1}$. The homomorphisms corresponding to caps and cups are between $V\otimes V^*$ (or $V^*\otimes V$) and $\mathbb{C}$. This is discussed in the next section.

Let us note that once $T\mapsto\varphi_T$ is constructed, it gives a link invariant. The invariant produced from $V=L(q)$ is the Jones polynomial. An advantage of the present construction is that it is much easier to compute the Jones polynomial from a link diagram (we just need to decompose the diagram into the composition of the generating tangles and write the corresponding composition of homomorphisms, see Picture 3.5).

3.3. **Duality.** Let V be a finite dimensional representation of U. We are going to define natural homomorphisms between $V\otimes V^*$ (and $V^*\otimes V$) and the trivial module $\mathbb{C}$. Recall that U acts on V^* via $\langle u\alpha, v\rangle = \langle \alpha, S(u)v\rangle$. Recall that S is given by $S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}$.

First of all, note that the natural isomorphism $V \cong V^{**}$ is not U-linear. Indeed, U acts on $V^{**} = V$ via $u \cdot v = S^2(u)v$, where in the right hand side we have the usual action on V. We get $S^2(u) = K^{-1}uK$ (it is enough to check this on the generators, where this straightforward). So $v \mapsto K^{-1}v$ is a U-module isomorphism $V \to V^{**}$.

The natural map $p : V^*\otimes V \to \mathbb{C}, \alpha\otimes v \mapsto \alpha(v)$ is U-linear. For example, let us check that p intertwines E, i.e., $p \circ E = 0$. We have

$$E(\alpha\otimes v) = \Delta(E)(\alpha\otimes v) = (E\otimes 1 + K\otimes E)(\alpha\otimes v) = E\alpha\otimes v + K\alpha\otimes Ev,$$
$$p(E(\alpha\otimes v)) = \langle E\alpha, v\rangle + \langle K\alpha, Ev\rangle = \langle \alpha, -K^{-1}Ev\rangle + \langle \alpha, K^{-1}Ev\rangle = 0.$$

The map $V^{**}\otimes V^* \to \mathbb{C}$ is U-linear hence $V\otimes V^* \to \mathbb{C}, v\otimes\alpha \mapsto \langle K^{-1}v, \alpha\rangle$ is U-linear.

Now let us get U-linear isomorphisms $\mathbb{C} \to V\otimes V^*, V^*\otimes V$. The former is the naive map: we can identify $V\otimes V^* \cong \mathrm{End}(V)$ via $(v\otimes\alpha).v' = \langle \alpha, v'\rangle v$ and the image of 1 in $V\otimes V^*$ is the identity map. This map is U-linear because the map $V\otimes V^*\otimes V \to V$ is U-linear. Similarly, we define $\mathbb{C} \to V^*\otimes V$ in such a way that the map $V^*\otimes V\otimes V^* \to V^*$ is U-linear: $1 \in \mathbb{C}$ goes to $K_{V^*}^{-1}$ under the natural identification $V^*\otimes V \cong \mathrm{End}(V^*)$.

We will use the notations ev_V for $V^*\otimes V \to \mathbb{C}$, ev_V^* for $V\otimes V^* \to \mathbb{C}$, coev_V for $\mathbb{C} \to V^*\otimes V$ and $\mathsf{coev}_V^* : \mathbb{C} \to V\otimes V^*$.

Example 3.1. Let us compute the four maps above for $V = L(q)$. Let v_1, v_2 be the natural basis of V and α_1, α_2 be the dual basis in V^*. Then $\mathsf{ev}_V(\sum_{i,j=1}^2 a_{ij}\alpha_i\otimes v_j) = a_{11}+a_{22}$, $\mathsf{ev}_V^*(\sum_{i,j=1}^2 b_{ij}v_i\otimes\alpha_j) = q^{-1}b_{11}+qb_{22}$. Further, $\mathsf{coev}_V^*(1) = v_1\otimes\alpha_1+v_2\otimes\alpha_2$, $\mathsf{coev}_V = q\alpha_1\otimes v_1 + q^{-1}\alpha_2\otimes v_2$. Note that $\mathsf{ev}_V\circ\mathsf{coev}_V = \mathsf{ev}_V^*\circ\mathsf{coev}_V^* = q+q^{-1}$.

These maps are assigned to cups and caps as shown in Picture 3.6.