LECTURE 11: SOERGEL BIMODULES

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Introduction

In this lecture we continue to study the category $\mathcal{O}_0$ and explain some ideas towards the proof of the Kazhdan-Lusztig conjecture.

We start by introducing projective functors $\mathcal{P}_i : \mathcal{O}_0 \to \mathcal{O}_0$ that act by $w \mapsto w(1+s_i)$ on $K_0(\mathcal{O}_0)$. Using these functors we produce a projective generator of $\mathcal{O}_0$.

In Section 2 we explain some of the work of Soergel that ultimately was used by Elias and Williamson to give a relatively elementary proof of the Kazhdan-Lusztig conjecture. In order to relate the category $\mathcal{O}_0$ to the Hecke algebra $\mathcal{H}_q(W)$ one needs to produce a graded lift of that category. In order to do that, Soergel constructed a functor $\mathcal{O}_0 \to \mathbb{C}[\mathfrak{h}]^{coW}$-mod, where $\mathbb{C}[\mathfrak{h}]^{coW}$ is the so called *coinvariant algebra*. He proved that this functor is fully faithful on the projective objects and has described the image of a projective generator that turns out to be a graded module. This gives rise to a graded lift of $\mathcal{O}_0$. Also these results of Soergel lead to the notion of Soergel (bi)modules that are certain (bi)modules over $\mathbb{C}[\mathfrak{h}]$. They are of great importance for modern Representation theory.

We finish by briefly describing some related constructions: Kazhdan-Lusztig bases for Hecke algebras with unequal parameters and multiplicities for rational representations of semisimple algebraic groups in positive characteristic.

1. Projective functors, II

1.1. **Functors $\mathcal{P}_i$.** Let $\alpha_1, \ldots, \alpha_n$ denote the simple roots. We want to define a projective functor $\mathcal{P}_i : \mathcal{O}_0 \to \mathcal{O}_0$. For this we pick $\lambda, \mu \in P$ such that $\lambda, \mu+\rho, \lambda-\mu$ are dominant and the only positive root vanishing on $\mu+\rho$ is α_i (so $\lambda+\rho$ lies inside the dominant Weyl chamber and $\mu+\rho$ lies on the wall corresponding to α_i). Let V be the irreducible module with highest weight $\lambda-\mu$. So we have functors $\varphi := \mathsf{pr}_\lambda(V \otimes \bullet) : \mathcal{O}_\mu \to \mathcal{O}_\lambda$ and its biadjoint $\varphi^* := \mathsf{pr}_\mu(V^* \otimes \bullet) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$.

Lemma 1.1. *The object $\varphi \circ \varphi^*(\Delta(w\cdot\lambda))$ admits a 2 step filtration by $\Delta(w\cdot\lambda), \Delta(ws_i\cdot\lambda)$, where the Verma with the smaller weight appears as the quotient and Verma with the larger weight appears as a sub.*

This is a special case of Proposition 2.3 in the previous lecture.

Now recall (Corollary 2.4 of Lecture 10) that there is an equivalence $\mathcal{O}_0 \xrightarrow{\sim} \mathcal{O}_\lambda$ with $\Delta(w\cdot 0) \mapsto \Delta(w\cdot\lambda)$. Transferring the functor $\varphi\circ\varphi^*$ to $\mathcal{O}_0$, we get the functor $\mathcal{P}_i$ we need. Note that $\mathcal{P}_i$ is exact (and, moreover, is self-adjoint). In particular, $\mathcal{P}_i$ induces an endomorphism of $K_0(\mathcal{O}_0) = \mathbb{Z}W$ to be denoted by $[\mathcal{P}_i]$.

Corollary 1.2. *We have $[\mathcal{P}_i]w = w(s_i+1)$.*

Remark 1.3. This may be viewed as a reason for the identification $K_0(\mathcal{O}_0) \cong \mathbb{Z}W$ (that is regarded as a right W-module). Indeed, we see that the generators s_i+1 of the group algebra $\mathbb{Z}W$ lift to endofunctors $\mathcal{P}_i$ of $\mathcal{O}_0$. This gives one of the first examples of *categorification.*

Remark 1.4. One can ask whether it is possible to lift the elements s_i to endofunctors of $\mathcal{O}_0$. The answer is yes if we are willing to replace $\mathcal{O}_0$ with the bounded derived category $D^b(\mathcal{O}_0)$. Then we can consider the *reflection functor* R_i given by the cone of the adjunction morphism $1 \to \mathcal{P}_i = \varphi \circ \varphi^*$. The functors R_i are self-equivalences of $D^b(\mathcal{O}_0)$ that give rise to an action of the braid group B_W on $D^b(\mathcal{O}_0)$. Recall that the group B_W is generated by the elements T_i with relations $T_iT_jT_i \ldots = T_jT_iT_j \ldots$ (m_{ij} factors).

1.2. **Projective objects in $\mathcal{O}_0$.** We start by recalling basics on projective objects in an abelian category, say $\mathcal{C}$. An object $P \in \mathcal{C}$ is called *projective* if the functor $\operatorname{Hom}_{\mathcal{C}}(P, \bullet)$ is exact. By $\mathcal{C}$-proj we denote the full subcategory of $\mathcal{C}$ consisting of projective objects ("full" means that the morphisms in $\mathcal{C}$-proj are the same as in $\mathcal{C}$). Note that this is an additive category that is closed under taking direct summands but, in general, has neither kernels nor cokernels.

We say that $\mathcal{C}$ has enough projectives if every object is a quotient of a projective. If all objects in $\mathcal{C}$ have finite length, the condition that $\mathcal{C}$ has enough projectives is equivalent to the condition that every simple $L \in \mathcal{C}$ has a *projective cover* P_L, i.e., an indecomposable projective object surjecting onto L (recall that an object is called *indecomposable* if it cannot be decomposed into a proper direct sum). If a projective cover exists, it is unique up to an isomorphism and has no Hom's to other simple objects. Every projective splits into the direct sum of projective covers.

We can recover $\mathcal{C}$ from $\mathcal{C}$-proj if $\mathcal{C}$ has enough projectives and all objects have finite length. For simplicity, assume, in addition, that $\mathcal{C}$ has finitely many simple objects. Let P be a projective generator of $\mathcal{C}$, i.e., a projective object that surjects onto any simple object. Then the functor $\operatorname{Hom}_{\mathcal{C}}(P, \bullet)$ is an equivalence of $\mathcal{C}$ and the category of right $\operatorname{End}_{\mathcal{C}}(P)$-modules. We note that $\operatorname{End}_{\mathcal{C}}(P)$ is a finite dimensional associative unital algebra. Conversely, if A is a finite dimensional associative unital algebra, then the category A-mod of finite dimensional A-modules has enough projectives, finitely many simples and all objects have finite length.

We are interested in $\mathcal{C} = \mathcal{O}_0$. As we have seen in Lecture 7, it has finitely many simples and all objects have finite length. Let us give an example of a projective.

Lemma 1.5. *The object $\Delta(0)$ is projective in $\mathcal{O}_0$.*

Proof. Recall that $\operatorname{Hom}_{\mathcal{O}_0}(\Delta(0), M) = \{m \in M_0 | \mathfrak{n}m = 0\}$. But 0 is the maximal weight of an object in $\mathcal{O}_0$ (because $w \cdot 0 \leqslant 0$ for all w). Since the action of $\mathfrak{n}$ increases weights, we see that $M_0^{\mathfrak{n}} = M_0$. The functor $M \mapsto M_0$ is exact and so $\Delta(0)$ is projective. □

Now let $w \in W$ have reduced expression $s_{i_1} \ldots s_{i_\ell}$. Let $\underline{w}$ denote the sequence $(s_{i_1}, \ldots, s_{i_\ell})$. Set $P_{\underline{w}} := \mathcal{P}_{i_\ell} \ldots \mathcal{P}_{i_1}\Delta(0)$.

Proposition 1.6. *The category $\mathcal{O}_0$ has enough projectives. Moreover, the object $P_{\underline{w}}$ decomposes into the direct sum of the projective cover of $L(w \cdot 0)$ (it appears with multiplicity 1) and of the projective covers of $L(u \cdot 0)$ with $u \prec w$ (in the Bruhat order).*

Below $P(w \cdot 0)$ denotes the projective cover of $L(w \cdot 0)$.

Proof. The functors $\mathcal{P}_i$ are self-adjoint and so map projectives to projectives. Hence $P_{\underline{w}}$ is projective. So it is enough to show that $\dim \operatorname{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w \cdot 0)) = 1$ and $\operatorname{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w' \cdot 0)) \neq 0$ implies $w' \preceq w$. The object $P_{\underline{w}}$ has a filtration with successive Verma quotients. Set $w_k = s_{i_1} \ldots s_{i_k}$ and $\underline{w}_k := (s_{i_1}, \ldots, s_{i_k})$. By induction on k (where the step follows from Lemma 1.1), we see that

(1) $\Delta(w_k \cdot 0)$ is a quotient of $P_{\underline{w}_k}$.

(2) All other Verma modules in a filtration of $P_{\underline{w}_k}$ have highest weights $u\cdot 0$, where u is obtained from w_k by removing some simple reflections (and hence $u \prec w_k$).

For $k=\ell$, (1) implies $\mathrm{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w\cdot 0)) \neq 0$, while (2) shows that $\mathrm{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w'\cdot 0)) \neq 0$ for $w' \neq w$ implies $w' \prec w$, and $\dim \mathrm{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w\cdot 0)) < 1$. □

Example 1.7. Consider the case of $\mathfrak{sl}_2$. Then $0 \to \Delta(0) \to P_s \to \Delta(-2) \to 0$. This exact sequence does not split (this is a part of the homework). So $P_s = P(-2)$. Applying $\mathcal{P}$ to $P(-2)$, we get $P(-2)^{\oplus 2}$.

2. Soergel (bi)modules

Our goal is to give some description of $\mathcal{O}_0$-proj and see that $\mathcal{O}_0$ is equivalent to A-mod, where A is graded. We will follow an approach by Soergel, [S1]. But first let us explain why we need graded algebras here.

2.1. **Graded lift.**

The first problem in proving the Kazhdan-Lusztig conjecture is that we do not know how to relate $\mathcal{O}_0$ to $\mathcal{H}_q(W)$: the Grothendieck group of $\mathcal{O}_0$ is $\mathbb{Z}W$, it does not "see" q, while it is impossible to define the Kazhdan-Lusztig basis without having q. This is remedied by considering "graded lifts".

Namely, let A be a finite dimensional algebra equipped with an algebra grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$. We can consider the category $\mathcal{C} := A$-mod. Or we can consider the category of *graded* A-modules. Its objects are finite dimensional graded A-modules, i.e., A-modules M equipped with an A-module grading $M = \bigoplus_{i\in\mathbb{Z}} M_i$ (this is a part of the structure). The morphisms in this category are the grading preserving homomorphisms of A-modules. Let us write $\tilde{\mathcal{C}}$ for the category of graded A-modules. It is an abelian category and we can consider its Grothendieck group $K_0(\tilde{\mathcal{C}})$.

The point is that $K_0(\tilde{\mathcal{C}})$ is not only an abelian group, but also a $\mathbb{Z}[q^{\pm 1}]$-module. For a graded A-module, we can consider the module $M\{d\}$ with shifted grading $M\{d\}_i := M_{i+d}$. We set $q^d[M] := [M\{d\}]$.

Not all A-modules admit a grading. However, we have the following lemma.

Lemma 2.1. *All indecomposable projective A-modules and all simple A-modules admit a grading. Furthermore, if an indecomposable A-module admits a grading, then the grading is unique up to a shift.*

We do not give a proof (it is actually based on the properties of algebraic groups – a basic observation here is that a grading on M gives rise to a $\mathbb{C}^\times$-action on M).

So fix some gradings on the simple A-modules $L_1,\ldots,L_k$. It follows that the simple graded A-modules are precisely $L_i\{d\}, i=1,\ldots,k, d\in\mathbb{Z}$. Hence $K_0(\tilde{\mathcal{C}})$ is a free $\mathbb{Z}[q^{\pm 1}]$-module with basis $[L_1],\ldots,[L_k]$. Besides $K_0(\mathcal{C}) = K_0(\tilde{\mathcal{C}})/(q-1)$.

Now let P be a projective generator of $\mathcal{O}_0$. Then $A := \mathrm{End}_{\mathcal{O}_0}(P)$ is a finite dimensional algebra and $\mathcal{O}_0 \cong A^{opp}$-mod (the category of right A-modules). The first step to prove the Kazhdan-Lusztig conjecture is to equip A with a grading (for a suitable choice of P).

2.2. **Structural results on $\mathcal{O}_0$-proj.**

The idea of Soergel, [S1], was to study the functor $\mathbb{V} := \mathrm{Hom}_{\mathcal{O}_0}(P(w_0\cdot 0), \bullet)$. The projective $P(w_0\cdot 0)$ plays a very special role, for example, it is the only indecomposable projective that is also injective. First, one needs to understand the target category for this functor, i.e., to compute the endomorphisms of $P(w_0\cdot 0)$. Let $\mathbb{C}[\mathfrak{h}]^W_+$ denote the ideal of all elements in $\mathbb{C}[\mathfrak{h}]^W$ without constant term. We write $\mathbb{C}[\mathfrak{h}]^{coW}$ (the "coinvariant algebra") for the quotient $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+)$, where we write $(\mathbb{C}[\mathfrak{h}]^W_+)=\mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]^W_+$. This is a graded algebra that is isomorphic to $\mathbb{C}W$ as a W-module.

Theorem 2.2. *We have* $\operatorname{End}_{\mathcal{O}_0}(P(w_0\cdot 0))\xrightarrow{\sim}\mathbb{C}[\mathfrak{h}]^{coW}$.

The functor $\mathbb{V}$ is very far from being an equivalence, for example, it kills all simples but $L(w_0\cdot 0)=\Delta(w_0\cdot 0)$. However, we have the following important result.

Theorem 2.3. *The functor* $\mathbb{V}$ *is fully faithful (induces an isomorphism of Hom spaces) on the projective objects.*

In the case of $\mathfrak{sl}_2$, these theorems can be verified directly (this is a part of the homework).

So, for a projective generator P of $\mathcal{O}_0$, we get $\operatorname{End}_{\mathcal{O}_0}(P)=\operatorname{End}_{\mathbb{C}[\mathfrak{h}]^{coW}}(\mathbb{V}(P))$.

Now let us compute $\mathbb{V}(\Delta(0))$ and understand what the functor $\mathcal{P}_i$ looks like on the level of $\mathbb{C}[\mathfrak{h}]^{coW}$-mod. Let $\mathbb{C}[\mathfrak{h}]^{s_i}$ denote the subalgebra of all s_i-invariant polynomials in $\mathbb{C}[\mathfrak{h}]$.

Theorem 2.4. *We have a functorial isomorphism* $\mathbb{V}(\mathcal{P}_i(M))\cong\mathbb{V}(M)\otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}}\mathbb{C}[\mathfrak{h}]$. *Moreover,* $\mathbb{V}(\Delta(0))\cong\mathbb{C}(=\mathbb{C}[\mathfrak{h}]/(\mathfrak{h}))$.

Note that $\mathbb{C}[\mathfrak{h}]$ is a graded $\mathbb{C}[\mathfrak{h}]^{s_i}$-module. So if $\mathbb{V}(M)$ is a graded $\mathbb{C}[\mathfrak{h}]^{coW}$-module, then $\mathbb{V}(\mathcal{P}_iM)$ gets graded (as a tensor product of graded modules). So the object $\mathbb{V}(P_{\underline{w}})$ gets graded. Moreover, $P:=\bigoplus_{\underline{w}}P_{\underline{w}}$ (we take one reduced expression per w) is a projective generator of $\mathcal{O}_0$ by Proposition 1.6.

Corollary 2.5. *The algebra* $A:=\operatorname{End}_{\mathcal{O}_0}(P)$ *is graded.*

Proof. By Theorem 2.3, we have $A=\operatorname{End}_{\mathbb{C}[\mathfrak{h}]^{coW}}(\mathbb{V}(P))$. The algebra $\mathbb{C}[\mathfrak{h}]^{coW}$ is graded and $\mathbb{V}(P)$ admits a grading. So $\operatorname{End}_{\mathbb{C}[\mathfrak{h}]^{coW}}(\mathbb{V}(P))$ has a natural grading. □

2.3. **Soergel modules and bimodules.** Consider the $\mathbb{C}[\mathfrak{h}]$-bimodule $\mathcal{B}_{s_i}:=\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}}\mathbb{C}[\mathfrak{h}]$. It is a graded bimodule (by the total degree), where, for convenience, we take $\deg\mathfrak{h}=2$. For a sequence $\underline{w}:=(s_{i_1},\ldots,s_{i_k})$ of simple reflections, we set $\mathcal{B}_{\underline{w}}:=\mathcal{B}_{s_{i_1}}\otimes_{\mathbb{C}[\mathfrak{h}]}\mathcal{B}_{s_{i_2}}\otimes_{\mathbb{C}[\mathfrak{h}]}\ldots\otimes_{\mathbb{C}[\mathfrak{h}]}\mathcal{B}_{s_{i_k}}$. This is a so called *Bott-Samelson* $\mathbb{C}[\mathfrak{h}]$-bimodule. Note that the left and right $\mathbb{C}[\mathfrak{h}]^W$-module structures on $\mathcal{B}_{\underline{w}}$ coincide.

Remark 2.6. The bimodules $\mathcal{B}_{\underline{w}}$ have a geometric meaning. Let $P_i, i=1,\ldots,n$ denote the minimal parabolic subgroup of G corresponding to the simple root α_i, its Lie algebra $\mathfrak{p}_i$ equals $\mathfrak{b}\oplus\mathbb{C}f_i$. Consider the product $P_{i_1}\times\ldots\times P_{i_k}$. On this product, the group B^k acts: $(b_1,\ldots,b_k).(p_1,\ldots,p_k)=(p_1b_1^{-1},b_1p_2b_2^{-1},\ldots,b_{k-1}p_kb_k^{-1})$. The quotient $P_{i_1}\times\ldots\times P_{i_k}/B^k$ is a so called Bott-Samelson variety, to be denoted by $\mathsf{BS}_{\underline{w}}$, since $P_{i_k}/B\cong\mathbb{P}^1$, the variety $\mathsf{BS}_{\underline{w}}$ is an iterated $\mathbb{P}^1$-bundle. Now suppose that $\underline{w}$ is a reduced expression of w. Note that we have a natural morphism $\mathsf{BS}_{\underline{w}}\to G/B$ (taking the product). One can show that the image is the Schubert subvariety $\overline{BwB}/B$ and that the morphism we consider is actually a resolution of singularities.

The group B still acts on $\mathsf{BS}_{\underline{w}}$ on the left and we can consider its equivariant cohomology $H^*_B(\mathsf{BS}_{\underline{w}})$. This is a module over $H^*_B(G/B)=\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^W}\mathbb{C}[\mathfrak{h}]$. In fact, $H^*_B(\mathsf{BS}_{\underline{w}})=\mathcal{B}_{\underline{w}}$. From here we deduce that $H^*(\mathsf{BS}_{\underline{w}})=\mathbb{C}\otimes_{\mathbb{C}[\mathfrak{h}]}\mathcal{B}_{\underline{w}}$, the module over $H^*(G/B)=\mathbb{C}[\mathfrak{h}]^{coW}$.

Definition 2.7. By a *Soergel bimodule* we mean any graded $\mathbb{C}[\mathfrak{h}]$-bimodule that appears as a graded direct summand of the bimodules of the form $\bigoplus_{\underline{w}}\mathcal{B}_{\underline{w}}\{d_{\underline{w}}\}$, where $d_{\underline{w}}\in\mathbb{Z}$ is a grading shift. A *Soergel module* is a graded direct summand in $\mathbb{C}\otimes_{\mathbb{C}[\mathfrak{h}]}B$, where B is a Soergel bimodule.

The motivation is as follows. The Soergel modules are precisely the images of the projectives in $\mathcal{O}_0$ under the functor $\mathbb{V}$. Taking the tensor product with $\mathcal{B}_{s_i}$ corresponds to the functor $\mathcal{P}_i$. So we should view Soergel bimodules as graded analogs of the projective functors.

The following result of Soergel classifies the indecomposable Soergel (bi)modules.

Theorem 2.8. *The indecomposable Soergel (bi)modules (up to a grading shift) are parameterized by W. More precisely, let $w \in W$, and $\underline{w} = (s_{i_1}, \ldots, s_{i_\ell})$ give a reduced expression of W. Then there is a unique indecomposable summand of $\mathcal{B}_{\underline{w}}$ depending only on w that does not occur in $\mathcal{B}_{\underline{u}}$, where $\ell(\underline{u}) < \ell$. Further, a Soergel bimodule B is indecomposable if and only if $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} B$ is indecomposable.*

The classification of the indecomposable Soergel modules can be deduced from their connection to the projective objects in $\mathcal{O}$. For bimodules, the claim is more complicated.

Example 2.9. Consider the bimodule $\mathcal{B}_s = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^s} \mathbb{C}[\mathfrak{h}]$. It is indecomposable. Indeed, $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathcal{B}_s = \mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^s_+)$ is indecomposable as a $\mathbb{C}[\mathfrak{h}]$-module and the graded Nakayma lemma implies that $\mathcal{B}_s$ is indecomposable. On the other hand, for $W = S_3$, the bimodule $\mathcal{B}_{(s,t,s)}$ is not indecomposable, it is the direct sum of $\mathcal{B}_s$ and $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$ with suitable grading shifts, both summands are indecomposable.

2.4. **Tensor structure and graded K_0.** By definition, $\mathsf{BS}_{\underline{w}} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathsf{BS}_{\underline{u}} = \mathsf{BS}_{\underline{wu}}$, where we write $\underline{wu}$ for the concatenation of $\underline{w}$ and $\underline{u}$. Because of this, Sbim is closed under taking tensor products over $\mathbb{C}[\mathfrak{h}]$. We write $B \cdot B'$ for $B \otimes_{\mathbb{C}[\mathfrak{h}]} B'$.

The category Sbim is not abelian (it does not have kernels and cokernels). But it is additive and is closed under taking direct summand. For such a category $\mathcal{C}$, we can define the *split* Grothendieck group, the quotient of the free group on the isomorphism classes of objects by $M = M' + M''$ if $M \cong M' \oplus M''$. We still denote this group by $K_0(\mathcal{C})$ (for an abelian category, the split Grothendieck group is huge and not useful at all, so this abuse of notation does not harm) and write $[M]$ for a class of M. A basis in $K_0(\mathcal{C})$ is formed by the (graded isomorphism classes of) indecomposable objects. Since we have the grading shifts on Sbim, the group $K_0(\mathsf{Sbim})$ is a $\mathbb{Z}[q^{\pm 1}]$-module. We can define a $\mathbb{Z}[q^{\pm 1}]$-algebra structure on $K_0(\mathsf{Sbim})$ by $[B] \cdot [B'] = [B \cdot B']$.

The following result is due to Soergel (a.k.a. Soergel's categorification theorem).

Theorem 2.10. *We have a $\mathbb{Z}[q^{\pm 1}]$-isomorphism $K_0(\mathsf{Sbim}) \cong \mathcal{H}_q(W)$ that sends $[\mathcal{B}_s]$ to $q^{-1}T_s + q^{-2}$.*

Sketch of proof. Both sides are free $\mathbb{Z}[q^{\pm 1}]$-modules of rank $|W|$. So the only thing that we need to prove is that $q[\mathcal{B}_s] - q^{-1}$ satisfies the relations for the elements T_s. We have relations of two kinds: quadratic relations $T_s^2 = 1 + (q - q^{-1})T_s$ and the braid relations $T_sT_tT_s \ldots = T_tT_sT_t \ldots$. The former are easy and we will check them, the latter are more complicated (the case $m_{st} = 2$ is still easy, the case of $m_{st} = 3$ follows from Example 2.9).

What we need to check is an isomorphism $\mathcal{B}_s \cdot \mathcal{B}_s = \mathcal{B}_s \oplus \mathcal{B}_s\{-2\}$. This is easily reduced to the case of $\mathfrak{sl}_2$. There $\mathcal{B}_s = \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]$ and

$$\mathcal{B}_s \cdot \mathcal{B}_s = (\mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]) \otimes_{\mathbb{C}[x]} (\mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]) = \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x].$$

As a $\mathbb{C}[x^2]$-bimodule, $\mathbb{C}[x]$ decomposes as $\mathbb{C}[x^2] \oplus \mathbb{C}[x^2]\{-2\}$ (the second summand is $x\mathbb{C}[x^2]$ and, by our convention, $\deg x = 2$). We conclude that $\mathcal{B}_s \cdot \mathcal{B}_s = \mathcal{B}_s \oplus \mathcal{B}_s\{-2\}$. ☐

2.5. **Kazhdan-Lusztig conjecture via Soergel bimodules.** A reasonable question is what are the classes of the indecomposable Soergel bimodules in $\mathcal{H}_q(W)$. Let $\underline{\mathcal{B}}_w$ denote the indecomposable summand in $\mathsf{BS}_{\underline{w}}\{\ell(w)\}$ that does not appear in $\mathsf{BS}_{\underline{u}}$ with $\ell(\underline{u}) < \ell(w)$. We want to describe the classes of $\underline{\mathcal{B}}_w$ (note that we have normalized the choice of grading).

It turns out that these classes are very closely related to the basis elements C_w. Namely, we have a ring involution $\bullet^*$ of $\mathcal{H}_q(W)$ defined on the generators by $T_s \mapsto T_s, q \mapsto -q^{-1}$ (this clearly preserves the relations).

Theorem 2.11. *We have* $[\underline{\mathcal{B}}_w] = C_w^*$.

This theorem can be shown to imply the Kazhdan-Lusztig conjecture (and determines the classes of $[P(w \cdot 0)] \in \mathbb{Z}W = K_0(\mathcal{O}_0)$). It was first proved by Soergel (using the geometric methods such as perverse sheaves). An alternative proof follows from the work of Elias and Williamson, see [EW1], and also a survey [EW2]. The main new ingredient of that work is a very clever emulation of Hodge theory in the context of Soergel bimodules.

3. Complements

3.1. **Hecke algebras with unequal parameters.** We have used the specialization of all Hecke algebra parameters v_s to q^2. One can consider more general specializations: v_s gets specialized to $q^{2L(s)}$, where $L : S \to \mathbb{Z}_{\geqslant 0}$ is a function that is constant on the conjugacy classes of reflections. The Kazhdan-Lusztig basis in the corresponding specialization is defined as before. However, these bases are much more mysterious. For example, let $P_w^u(q)$ be the coefficient of T_u in C_w. In the equal parameter case it is known that $(-1)^{\ell(w)-\ell(u)} P_w^u(q)$ has nonnegative coefficients. This is not known in general.

3.2. **Multiplicities for algebraic groups.** The Kazhdan-Lusztig basis in a suitable Hecke algebra controls the characters of irreducible $G_{\mathbb{F}}$-modules $L(\lambda)$, where $G_{\mathbb{F}}$ is a semisimple algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic p, where p is very large comparing to the rank of $G_{\mathbb{F}}$ (the dimension of the maximal torus). The Hecke algebra is taken for the affine Weyl group $W^{aff} = W \ltimes Q$, where W is the Weyl group of G and Q is the root lattice, i.e., the group in $\mathfrak{h}^*$ generated by the simple roots, compare to Problem 5 in Homework 2. We refer to Section 2 of [F] for details.

References

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