LECTURE 1: REPRESENTATIONS OF SYMMETRIC GROUPS, I

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1. Introduction

In this lecture we start to study the representation theory of the symmetric groups S_n over $\mathbb{C}$. Let us summarize a few things that we already know.

0) A representation of S_n is the same thing as a representation of the group algebra $\mathbb{C}S_n$.

1) As with all finite groups, any representation of S_n over $\mathbb{C}$ is completely reducible.

2) The number of irreducible representations of S_n coincides with the number of conjugacy classes.

The conjugacy classes are in a natural bijection with partitions of n. Namely, we take an element $\sigma \in S_n$ and decompose it into the product of disjoint cycles. The lengthes of cycles form a partition of n that is independent of the choice of σ in the conjugacy class. We assign this partition to the conjugacy class of σ.

We would like to emphasize that 2) does not establish any preferred bijection between the irreducible representations of S_n and the partitions of n. To establish such a bijection is our goal in this part. We will follow a "new" approach to the representation theory of the groups S_n due to Okounkov and Vershik, [OV]. Our exposition follows [K, Section 2]. For a "traditional" approach based on Young symmetrizers, the reader is welcome to consult [E] or [F].

2. Inductive approach

A key observation is that symmetric groups for different n are embedded into one another: $\{1\} = S_1 \subset S_2 \subset S_3 \ldots \subset S_{n-1} \subset S_n \subset \ldots$, where we view S_{n-1} as the subgroup of S_n fixing $n \in \{1, \ldots, n\}$. We could try to use "induction", i.e., to study the irreducible representations of S_n by restricting them to S_{n-1}. In fact, this naive idea does not quite work, but this is our starting point.

2.1. **Centralizer $Z_B(A)$ and restrictions of representations.** We start with the following question: given a finite dimensional semisimple associative algebra A and its semisimple subalgebra B, understand the restriction of $V \in \operatorname{Irr}(A)$ (the set of isomorphism classes of finite dimensional irreducible A-modules) to B. The answer to this question is controlled by the subalgebra $Z_B(A) \subset A$ (the centralizer of B in A) defined by $Z_B(A) = \{a \in A | ba = ab, \forall b \in B\}$. More precisely, we have the following fact, where, recall that $\operatorname{Hom}_B(U, V)$ stands for the space of B-module homomorphisms $U \to V$.

Lemma 2.1. *We have an isomorphism* $Z_B(A) = \bigoplus_{U,V} \operatorname{End}(\operatorname{Hom}_B(U, V))$*, where the sum is taken over all pairs* $(U, V) \in \operatorname{Irr}(B) \times \operatorname{Irr}(A)$ *satisfying* $\operatorname{Hom}_B(U, V) \neq \{0\}$.

In other words, the algebra $Z_B(A)$ is semisimple and the irreducible $Z_B(A)$-modules are precisely the nonzero multiplicity spaces $\operatorname{Hom}_B(U, V)$.

Proof. Since A is semisimple, it can be identified with $\bigoplus_{V\in \mathrm{Irr}(A)}\mathrm{End}(V)$, see Proposition 4.2 from Lecture 0. In this realization, we have $Z_B(A)=\bigoplus_{V\in\mathrm{Irr}(A)}\mathrm{End}_B(V)$. Recall from Lecture 0, Section 3.4, that

$$\mathrm{End}_B(V)=\bigoplus_U \mathrm{End}(\mathrm{Hom}_B(U,V)),$$

where the summation is taken over all $U\in\mathrm{Irr}(B)$ such that $\mathrm{Hom}_B(U,V)\neq\{0\}$. $\square$

Here is a corollary of this lemma that will be very useful for us in what follows.

Corollary 2.2. *The following two conditions are equivalent:*

(1) *For any* $U\in\mathrm{Irr}(B), V\in\mathrm{Irr}(A)$*, we have* $\dim\mathrm{Hom}_B(U,V)\leqslant 1$.
(2) $Z_B(A)$ *is commutative.*

Proof. The algebra $Z_B(A)=\bigoplus_{U,V}\mathrm{End}(\mathrm{Hom}_B(U,V))$ is commutative if and only if the summands are. For a nonzero vector space W, $\mathrm{End}(W)=\mathrm{Mat}_{\dim W}(\mathbb{C})$ is commutative if and only if $\dim W=1$. This implies that (1) $\Leftrightarrow$ (2). $\square$

What this corollary gives us is that if (2) is satisfied, then any irreducible A-module uniquely decomposes into the sum of irreducible B-modules, meaning that the summands are uniquely determined as subspaces.

Remark 2.3. It is instructive to describe the structure of a $Z_B(A)$-module on $\mathrm{Hom}_B(U,V)$ without referring to the decomposition $A=\bigoplus\mathrm{End}(V)$. Let $z\in Z_B(A)$ and $\varphi\in\mathrm{Hom}_B(U,V)$. We define $z\cdot\varphi\in\mathrm{Hom}_B(U,V)$ by $[z\cdot\varphi](u)=z\cdot\varphi(u)$, for all $u\in U$. To check that this is well defined and is compatible with the previous module structure (see Section 3.4 of Lecture 0) is left to the reader.

2.2. **The structure of** $Z_m(n)$**.** We take $A=\mathbb{C}S_n$ and $B=\mathbb{C}S_m$, where $m<n$. We write $Z_m(n)$ for $Z_B(A)$. We are interested in finding generators for the algebra $Z_m(n)$.

Theorem 2.4. *The algebra* $Z_m(n)$ *is generated by the subalgebra* $Z_m(m)$*, the subgroup* $S_{[m+1,n]}\subset S_n$ *(fixing* $1,\ldots,m$ *and permuting* $m+1,\ldots,n$*) and the* Jucys-Murphy elements $L_k:=\sum_{i=1}^{k-1}(ik)$*, where* k *ranges from* $m+1$ *to* n.

Note that $L_1=0$.

Proof. The proof is in several steps.

Step 1. Let $H\subset G$ be finite groups. Then we have a basis in $Z_{\mathbb{C}H}(\mathbb{C}G)$ indexed by the H-conjugacy classes in G. Namely, to a class c we assign an element $b_c\in Z_{\mathbb{C}H}(\mathbb{C}G)$ given by $z_c:=\sum_{g\in c}g$. This is a straightforward generalization of Proposition 5.2 in Lecture 0.

Step 2. Let $G=S_n, H=S_m$. Recall that the G-conjugacy classes in G are parameterized by cycle types, e.g., in S_6 we have a conjugacy class $(***)(**)$. The H-conjugacy classes are parameterized by class types with marked elements $m+1,\ldots,n$, e.g., we have the following S_4-conjugacy classes in S_6: $(56*)(**),(65*)(**),(6**)(5*),(***)(56)$, etc. Note that $b_{(*k)}=L_k-\sum_{j=m+1}^{k-1}(jk)$ for $k>m$. In particular, $L_k\in Z_m(n)$. To a class c we assign its *degree* $\deg c$ that, by definition, is equal to the number of elements in $\{1,\ldots,n\}$ moved by an element in c (this is independent of the choice of the element).

Step 3. Let A be the subalgebra in $\mathbb{C}S_n$ generated by $Z_m(m),S_{[m+1,n]},L_{m+1},\ldots,L_n$. It is easy to see that $A\subset Z_m(n)$. To prove the opposite inclusion, assume that $b_c\notin A$ for some c. We pick c of minimal degree with this property. In the next four steps we will arrive at a contradiction.

Step 4. Assume, first, that c has more than one cycle of length at least 2. Break c into the union of two cycle types c', c'', e.g., if $c = (6**)(5*)$, then we can take $c' = (6**), c'' = (5*)$. Note that

$$b_{c'}b_{c''} = \alpha b_c + \sum_{c_0, \deg c_0 < \deg c} \alpha_{c_0} b_{c_0},$$

where $\alpha > 0$. By our inductive assumption, $b_{c_0} \in A$ and $b_{c'}b_{c''} \in A$. So $b_c \in A$, which contradicts the choice of c.

Step 5. Now let us pick a cycle $(i_1, i_2, \ldots, i_k) \in S_n$ and consider the product $(i_1, \ldots, i_k)(i_s, j)$. If $j \notin \{i_1, \ldots, i_k\}$, then we get $(i_1, \ldots, i_s, j, i_{s+1}, \ldots, i_k)$. If $j \in \{i_1, \ldots, i_k\}$, then $(i_1, \ldots, i_k)(i_k j)$ either splits into the product of two cycles of total degree k or is a cycle of degree $k-1$.

Step 6. Now suppose that the cycle in c has both an element from $\{1, \ldots, m\}$ (does not matter which, we denote it by $*$) and $k \in \{m+1, \ldots, n\}$. We may assume that k is right after $*$ in the cycle. Let c' denote the cycle obtained from c by deleting k. Then $b_{c'}L_k = \alpha b_c + \sum_{c_0} \alpha_{c_0} b_{c_0}$, where the summation is over c_0 that are products of two disjoint cycles with $\deg c_0 = \deg c$ or have $\deg c_0 < \deg c$. This is a consequence of Step 5, as the left hand side is the sum of products of pairs of cycles that share a common element, k. Similarly to Step 4, we arrive at a contradiction with the choice of c.

Step 7. So either the elements in the only cycle of c are all from $\{1, \ldots, m\}$, in which case $b_c \in Z_m(m)$, or are all from $\{m+1, \ldots, n\}$, in which case $b_c \in S_{[m+1,n]}$. Contradiction. $\square$

Corollary 2.5. *The following is true.*

1. *$Z_m(m)$ lies in the center of $Z_m(n)$.*
2. *The algebra $Z_{n-1}(n)$ is commutative.*

Proof. The algebra $Z_m(n)$ commutes with $\mathbb{C}S_m$ and $Z_m(m) \subset Z_m(n) \cap \mathbb{C}S_m$. So $Z_m(m)$ is in the center of $Z_m(n)$.

The algebra $Z_{n-1}(n)$ is generated by $Z_{n-1}(n-1)$ and L_n. Since the former is central, the algebra $Z_{n-1}(n)$ is commutative. $\square$

3. Basis and weights

We will use Corollary 2.5 to construct a basis in $\bigoplus_{V \in \mathrm{Irr}(S_n)} V$ and encode elements of this basis with n-tuples of complex numbers to be called *weights*.

3.1. **Branching graph.**

Basis elements will be labelled by paths in a graph that is called the branching graph for the symmetric groups. The vertices of this graph will be $\bigsqcup_{n \geqslant 1} \mathrm{Irr}(S_n)$. We draw a single arrow between $V^{n-1} \in \mathrm{Irr}(S_{n-1}), V^n \in \mathrm{Irr}(S_n)$ if $\mathrm{Hom}_{S_{n-1}}(V^{n-1}, V^n)$ has dimension 1 (by Corollary 2.5 the only other option is 0). There are no other edges.

Paths in the branching graph label bases in Hom spaces. For vertices $V^m \in \mathrm{Irr}(S_m), V^n \in \mathrm{Irr}(S_n)$ with $m < n$, denote by $\mathsf{Path}(V^m, V^n)$ the set of paths from V^m to V^n.

Lemma 3.1. *There is a basis in* $\mathrm{Hom}_{S_m}(V^m, V^n)$ *indexed by* $\mathsf{Path}(V^m, V^n)$.

Proof. We have $V^n = \bigoplus_{V^{n-1}, V^{n-1} \to V^n} V^{n-1}$. Now decompose V^{n-1} into the sum of irreducible representations of S_{n-2}. Plugging this decomposition into the sum above, we get

$$V^n = \bigoplus_{V^{n-2}, P \in \mathsf{Path}(V^{n-2}, V^n)} V_P^{n-2},$$

where V^{n-2}_P denotes the copy of V_n embedded into V^n via $V^{n-2} \hookrightarrow V^{n-1} \hookrightarrow V^n$, where $P = V^{n-2} \to V^{n-1} \to V^n$. We continue in this manner and get

$$V^n = \bigoplus_{V^m, P \in \mathsf{Path}(V^m, V^n)} V^m_P.$$

Let $\varphi_P \in \mathrm{Hom}_{S_m}(V^m, V^n)$ be the embedding of $V^m \xrightarrow{\sim} V^m_P \subset V^n$. We see that $\varphi_P, P \in \mathsf{Path}(V^m, V^n)$, is a basis in $\mathrm{Hom}_{S_m}(V^m, V^n)$. □

Remark 3.2. Note that the element φ_P is defined uniquely up to proportionality. Also note that if $P_2 \in \mathsf{Path}(V^k, V^m), P_1 \in \mathsf{Path}(V^m, V^n)$, then $\varphi_{P_1} \circ \varphi_{P_2}$ is proportional to $\varphi_{P_1 P_2}$, where $P_1 P_2 \in \mathsf{Path}(V^k, V^n)$ is the concatenation of P_1 and P_2.

3.2. **Basis and weights.** If in Lemma 3.1 we take $m = 1$, we will get a basis in $\mathrm{Hom}_{S_1}(V^1, V^n) = \mathrm{Hom}(\mathbb{C}, V^n) = V^n$, we will write v_P for φ_P in this case. By the construction, if $P = V^1 \to V^2 \to \ldots \to V^n$, then v_P lies in V^1 uniquely embedded into V^2 that is uniquely embedded into V^3, etc.

Lemma 3.3. *The following is true.*

1. *The vector v_P is an eigenvector for all Jucys-Murphy elements $L_k, k = 1, \ldots, n$.*
2. *The eigenvalue of L_k on v_P depends only on the V^{k-1} and V^k components in $P = V^1 \to V^2 \to \ldots \to V^n$.*

We postpone the proof a little bit, to give a definition and an example.

Definition 3.4. Define the *weight* $w_P = (w_1, \ldots, w_n) \in \mathbb{C}^n$ of the path P (or of the basis vector v_P) by $L_k v_P = w_k v_P, k = 1, \ldots, n$.

Example 3.5. Consider the reflection representation R^n of S_n. It can be realized as the submodule $\{(x_1, \ldots, x_n) | x_1 + \ldots + x_n = 0\}$ in the permutation representation $\mathbb{C}^n$ of S_n. The restriction of R^n to S_{n-1} decomposes as $R^{n-1} \oplus T^{n-1}$, where we write T^{n-1} for the trivial representation of S_{n-1}. The copy of R^{n-1} is realized as $\{(x_1, \ldots, x_n) | x_1 + \ldots + x_{n-1} = 0, x_n = 0\}$, while the copy of T^{n-1} is spanned by $(1, \ldots, 1, 1-n)$. The paths indexing the basis in R^n are

$$P_m := T^1 \to T^2 \to T^{m-1} \to R^m \to R^{m+1} \to \ldots \to R^n, m = 2, \ldots, n.$$

The corresponding basis vector is $v_{P_m} = (1, \ldots, 1, 1-m, 0, \ldots, 0)$. The weight w_{P_m} equals $(0, 1, \ldots, m-2, -1, m-1, \ldots, n-2)$.

Proof of Lemma 3.3. Note that v_P lies in the unique copy of V^{k-1} in V^k. It is enough to check that L_k acts on that copy of V^{k-1} by a scalar (that depends only on V^{k-1}, V^k because $L_k \in \mathbb{C}S_k$). But L_k commutes with $\mathbb{C}S_{k-1}$ and so the operator of multiplication by L_k gives an element in $\mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^k)$. Since the dimension of the latter space is 1, the multiplication by L_{k-1} gives an endomorphism of the S_{k-1}-module V^{k-1}. This endomorphism is scalar by the Schur lemma. □

3.3. **Maximal commutative subalgebra.** A natural question to ask at this point is: can two different paths $P \in \mathsf{Path}(V^1, V^n), P' \in \mathsf{Path}(V^1, V'^n)$ give the same weight? Here we will see that the answer is "no": a weight determines a path uniquely.

Consider the subalgebra $A \subset \mathbb{C}S_n$ consisting of all elements a such that all v_P are eigenvectors for a. In other words, if we identify $\mathrm{End}(V^n)$ with $\mathrm{Mat}_{\dim V^n}(\mathbb{C})$ using the basis $v_P, P \in \mathsf{Path}(V^1, V^n)$, and $\mathbb{C}S_n$ with $\bigoplus_{V^n \in \mathrm{Irr}(S_n)} \mathrm{End}(V^n)$, then A is the direct sum of the subalgebras of diagonal matrices in $\mathrm{Mat}_{\dim V^n}(\mathbb{C})$. Note that A is a maximal commutative subalgebra in $\mathbb{C}S_n$.

There are two alternative descriptions of A.

Proposition 3.6. *The following subalgebras of $\mathbb{C}S_n$ coincide.*

(i) *A introduced above.*
(ii) *A' generated by $Z_k(k), k = 1, \dots, n$.*
(iii) *A'' generated by $L_1, \dots, L_n$.*

Proof. We will prove that $A \subset A', A' \subset A''$, and $A'' \subset A$.

Proof of $A \subset A'$. We have a basis in A labelled by the paths $P \in \mathsf{Path}(V^1, V^n)$, where V^n runs over $\mathrm{Irr}(S_n)$. Namely, define e_P by $e_P v_{P'} = \delta_{PP'} v_{P'}$ (i.e., e_P is the diagonal matrix element corresponding to P). Let $P = V^1 \to V^2 \to \dots \to V^n$.

Define $e_{V^m} \in \mathbb{C}S_m$ as the identity in the summand $\mathrm{End}(V^m)$ of $\mathbb{C}S_m = \bigoplus_{U \in \mathrm{Irr}(S_m)} \mathrm{End}(U)$ and zero in all other summands. This element is central, in other words, $e_{V^m} \in Z_m(m)$. Now consider the product $e_{V^1} e_{V^2} \dots e_{V^n}$ and its action on $\bigoplus_{U^n \in \mathrm{Irr}(S_n)} U^n$. Applying e_{V^n} we project to the summand V^n. Applying $e_{V^{n-1}}$ next, we project to the summand V^{n-1} inside V^n. And so on. From the construction of the element v_P, we conclude that $e_{V^1} \dots e_{V^n}$ coincides with e_P. Since $e_{V^1} \dots e_{V^n} \in A'$, we see that $e_P \in A'$, and we are done.

Proof of $A' \subset A$. We prove this by induction on n: suppose that $Z_1(1) = \mathbb{C}, \dots, Z_{n-1}(n-1)$ lie in the subalgebra generated by $L_1, \dots, L_{n-1}$. Note that $Z_n(n) \subset Z_{n-1}(n)$. By Theorem 2.4, $Z_{n-1}(n)$ is generated by $Z_{n-1}(n-1)$ and L_n. So $Z_n(n) \subset A''$ and hence $A' \subset A''$.

Proof of $A'' \subset A$. By Lemma 3.3, every v_P is an eigenvector for L_k. So $L_k \in A$ for any k. The inclusion $A'' \subset A$ follows. $\square$

Corollary 3.7. *If $P \neq P'$, then $w_P \neq w_{P'}$.*

Proof. If $w_P = w_{P'}$, then every element $a \in A''$ acts on $v_P, v_{P'}$ with the same eigenvalue. But $e_P \in A''$ obviously does not have this property. $\square$

3.4. **Road map.** Let $\mathsf{Wt}(n)$ denote the set of all possible weights, this is a subset of $\mathbb{C}^n$. On $\mathsf{Wt}(n)$ we have an equivalence relation: we say that $w_P \sim w_{P'}$, if P, P' lead to the same irreducible V^n. What we need to do to classify $\mathrm{Irr}(S_n)$ is to solve the following two problems:

a) Describe $\mathsf{Wt}(n)$.
b) Determine the equivalence relation $\sim$ on $\mathsf{Wt}(n)$.

This will be done in the next lecture.

References

[E] P. Etingof et al. *Introduction to representation theory.* Student Mathematical Library, 59. American Mathematical Society, Providence, RI, 2011. arXiv:0901.0827.

[F] W. Fulton. *Young tableaux. With applications to representation theory and geometry.* London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[K] A. Kleshchev. *Linear and projective representations of symmetric groups.* Cambridge Tracts in Mathematics, 163. Cambridge University Press, Cambridge, 2005.

[OV] A. Okounkov, A. Vershik. *A new approach to representation theory of symmetric groups.* Selecta Math. (N.S.) 2 (1996), no. 4, 581605.