RCA, PROBLEM SET 1

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0.1. Check that the formulas

$$x \mapsto x, w \mapsto w, y \mapsto D_y := \partial_y + \sum_{s\in S} \frac{c(s)\langle \alpha_s, y\rangle}{\alpha_s}(s-1), \quad x \in \mathfrak{h}^*, w \in W, y \in \mathfrak{h},$$

define an algebra homomorphism $H_c(W,\mathfrak{h}) \to D(\mathfrak{h}^{reg})\#W$ (e.g., following the sketch given in the lecture).

0.2. Check that the Euler element $h = \sum_{i=1}^n x_i y_i - \sum_{s\in S} c(s)s$ satisfies

$$[h,x] = x, [h,w] = 0, [h,y] = -y.$$

0.3. Let $\mathcal{C}$ be a highest weight category. Show that the standard object Δ_L is the projective cover of L in the Serre subcategory of $\mathcal{C}$ spanned by $L' \leqslant L$.

0.4. Show that in $\mathcal{O}_c$ we have the following (we don't know at this point that $\mathcal{O}_c$ is a highest weight category):

$$\begin{aligned}
&\operatorname{Hom}(\Delta_c(\tau), \Delta_c(\tau')) \neq 0 \Rightarrow \tau \leqslant_c \tau',\\
&\operatorname{End}(\Delta_c(\tau)) = \mathbb{C},\\
&\operatorname{Ext}^1(\Delta_c(\tau), \Delta_c(\tau')) \neq 0 \Rightarrow \tau <_c \tau'.
\end{aligned}$$

0.5. Prove that the map $[\tau] \mapsto [\Delta_c(\tau)]$ is an isomorphism $K_0(W\text{-rep}) \xrightarrow{\sim} K_0(\mathcal{O}_c)$. Furthermore, prove that two objects in $\mathcal{O}_c$ with the same character have the same classes in $K_0(\mathcal{O}_c)$.

0.6. Let $M_1, M_2 \in \mathcal{O}_c$ be such that $M_1 \oplus M_2$ has a Verma filtration. Show that both M_1, M_2 have Verma filtrations (again, at this point we still don't know that $\mathcal{O}_c$ is highest weight).

0.7. Let a parameter c be *generic* in the sense that $\tau \leqslant_c \tau'$ implies $\tau = \tau'$. Show that the category $\mathcal{O}_c$ is semisimple and that $P_c(\tau) = \Delta_c(\tau) = L_c(\tau)$.