Introduction to universal enveloping algebras & Verma modules

1) Preliminaries on s/simple Lie algebras
2) Universal enveloping algebras and Harish-Chandra isomorphism
3) Simple quotients of and homomorphisms between Verma modules
4)* What's to follow in the seminar

1) $\mathfrak{g}$ s/simple Lie algebra/$\mathbb{C}$, G conn'd alg'c gr'p w. $\mathrm{Lie}(G)=\mathfrak{g}$
$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ - Borel & Cartan subalgebras
$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$, where R is a root system
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, $\mathfrak{n}_+ := \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ (R^+ is the positive roots, sometimes write $\alpha>0$ for $\alpha\in R^+$), $\mathfrak{n}_- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha \rightsquigarrow \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (triang. decomp'n)

• generators: $S \subset R^+$ - simple roots, $\alpha\in S \rightsquigarrow e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha = [e_\alpha, f_\alpha]$. Often we number the simple roots: $\alpha_1, \dots, \alpha_r$ ($r = \dim \mathfrak{h}$) & write e_i, f_i, h_i for $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i}$. Of course $h_i = \alpha_i^\vee$ (the coroot)

Some relations: $[e_i, f_j] = \delta_{ij} h_j$, $[h_i, e_j] = \langle \alpha_j, \alpha_i^\vee \rangle e_j$, $[h_i, f_j] = -\langle \alpha_j, \alpha_i^\vee \rangle f_j$. If we add Serre relations (we don't need them now) we'll get a complete set of rel'ns for $\mathfrak{g}$ (*)

(*) Note that the e_i's generate $\mathfrak{n}_+$ (and the f_i's generate $\mathfrak{n}_-$)

• Weyl group: Let $H \subset G$ be connected w. $\mathrm{Lie}(H) = \mathfrak{h}$ (a.k.a. max torus). Then $W = N_G(\mathfrak{h})/H$ is called the Weyl gr'p. It faithfully acts on $\mathfrak{h}$ so we view it as a subgroup of $GL(\mathfrak{h})$. For $\alpha \in R$ we can consider the reflection $s_\alpha \in GL(\mathfrak{h})$: $s_\alpha(x) = x - \langle \alpha, x\rangle \alpha^\vee$, then $s_\alpha \in W$. We write s_i for s_{α_i}. Then W is a Coxeter group w. generators $s_1, \dots, s_r$

2.1) Def'n of $\mathcal{U}(\mathfrak{g})$

Let $\mathfrak{g}$ be a Lie alg'a (for simplicity, $\dim \mathfrak{g} < \infty$)
$\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y] \mid x, y \in \mathfrak{g})$ - assoc. alg'a w. Lie alg. homomorphism $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$. Note that, by definition, $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is an initial object in the category of Lie algebra homomorphisms $\mathfrak{g} \to A$, where A is an associative alg'a (morphism from $\mathfrak{g} \to A$

to $\mathfrak{g} \to B$ is an algebra homomorphism $A \to B$ making the obvious diagram commutative). In particular, a representation of $\mathfrak{g}$ is the same thing as a representation of $U(\mathfrak{g})$.

2.2) Filtrations & PBW theorem

The tensor algebra $T(\mathfrak{g})$ is graded (w. $\deg \mathfrak{g} = 1$). The 2-sided ideal $J = (x\otimes y - y\otimes x - [x,y] \mid x, y \in \mathfrak{g})$ is not homogeneous. Take the assoc'd graded of J (= linear span of top degree terms in the elements of J) denoted by $\mathrm{gr}\, J$. Note that $x\otimes y - y \otimes x \in \mathrm{gr}\, J$.

The algebra $U(\mathfrak{g}) = T(\mathfrak{g})/J$ inherits an algebra filtration from $T(\mathfrak{g})$ and we can consider the associated graded algebra $\mathrm{gr}\, U(\mathfrak{g})$. Of course, $\mathrm{gr}\, U(\mathfrak{g}) = T(\mathfrak{g})/\mathrm{gr}\, J$. Note that $S(\mathfrak{g}) = T(\mathfrak{g})/(x\otimes y - y\otimes x) \twoheadrightarrow T(\mathfrak{g})/\mathrm{gr}\, J = \mathrm{gr}\, U(\mathfrak{g})$.

Thm 1 (PBW) The epimorphism $S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}\, U(\mathfrak{g})$ is an iso.

Cor 1: Let $x_1, \dots x_n \in \mathfrak{g}$ be a basis. Then the ordered monomials $x_1^{d_1} \dots x_n^{d_n}$ ($d_i \geq 0$) form a basis in $U(\mathfrak{g})$.

Cor 2: Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a subalgebra. Then $U(\mathfrak{g})$ is a free left $U(\mathfrak{g}_0)$-module (and a free right $U(\mathfrak{g}_0)$-module) - Exercise

Cor 3 (triangular decomp'n) Let $\mathfrak{g}$ be s/simple. Then the map $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \longrightarrow U(\mathfrak{g})$, $a\otimes b \otimes c \mapsto abc$ is an iso of vector spaces.

We can also deduce some algebraic properties of $U(\mathfrak{g})$ from Thm 1

Prop 1: Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then

1) $U(\mathfrak{g})$ is left (and right) Noetherian
2) $U(\mathfrak{g})$ has no zero divisors

Proof: Note that $S(\mathfrak{g})$ is Noetherian & has no zero divisors. We have a general fact that if A is a $\mathbb{Z}_{\geq 0}$-filt'd assoc algebra such that $\mathrm{gr}\, A$ is Noetherian w/o zero divisors, then so is A. Now use Thm 1 $\square$

Rem'k: Note that G acts on $U(\mathfrak{g})$ preserving the filtration (the action is induced from $T(\mathfrak{g})$). The group G also acts on $S(\mathfrak{g})$

Both actions are by algebra automorphisms and the PBW isomorphism is G-equiv't.

2.3) Center of $\mathcal{U}(\mathfrak{g})$ & HC isomorphism ($\mathfrak{g}$ is s/simple from now on)

Recall that by the center of an associative algebra A one means $Z(A) := \{z \in A \mid za = az \ \forall a \in A\}$. For $A = \mathcal{U}(\mathfrak{g})$, the center has an equivalent alternative description

Lem 1: $Z(\mathcal{U}(\mathfrak{g})) = \mathcal{U}(\mathfrak{g})^G$

Proof: $\mathfrak{g}$ generates $\mathcal{U}(\mathfrak{g})$, so $z \in Z(\mathcal{U}(\mathfrak{g})) \Leftrightarrow [x,z]=0 \ \forall x \in \mathfrak{g}$. But note that the $\mathfrak{g}$-action on $\mathcal{U}(\mathfrak{g})$ given by $x \mapsto [x,\cdot]$ is the differential of the G-action so $[x,z]=0 \ \forall x \in \mathfrak{g} \Leftrightarrow z \in \mathcal{U}(\mathfrak{g})^G$ $\square$

To describe $Z(\mathcal{U}(\mathfrak{g}))$ we'll need a so called ρ-shifted W-action on $\mathfrak{h}^*$. Set $\rho = \frac{1}{2}\sum_{\alpha > 0} \alpha \in \mathfrak{h}^*$ and, for $w \in W$, $\lambda \in \mathfrak{h}^*$ define $w \cdot \lambda = w(\lambda+\rho) - \rho$. This defines an action of W on $\mathfrak{h}^*$ by affine transformations (w. fixed point $-\rho$). The associated linear action is the standard action $W \curvearrowright \mathfrak{h}^*$.

Now consider the algebra $\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$ of invariants in $\mathbb{C}[\mathfrak{h}^*]$ for the shifted W-action.

Thm 2 (Harish-Chandra) We have a natural isomorphism $Z(\mathcal{U}(\mathfrak{g})) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$

We'll explain what "natural" means in the proof. The steps of the proof are as follows

I: Construct a homomorphism $Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}[\mathfrak{h}^*]$ (using the triangular decomp'n of $\mathcal{U}(\mathfrak{g})$)

II: Show that the image is contained in $\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$ (using Verma modules and some homomorphisms between those)

III: Show that the homomorphism is injective and its image coincides with $\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$ (using the equality $Z(\mathcal{U}(\mathfrak{g})) = \mathcal{U}(\mathfrak{g})^G$ and the Chevalley restriction Thm)

2.4) Homomorphism $Z(\mathcal{U}(\mathfrak{g})) \xrightarrow{\varphi} \mathbb{C}[\mathfrak{h}^*]$

Set $\mathcal{U}(\mathfrak{g})_0 := \mathcal{U}(\mathfrak{g})^H$. In other words, if $x_1, \dots x_n$ is a weight basis in $\mathfrak{g}$ with weights $\lambda_1, \dots \lambda_n \in R \sqcup \{0\}$, then $\mathcal{U}(\mathfrak{g})_0$ is the span of monomials $x_1^{d_1} \dots x_n^{d_n}$ w $\sum d_i \lambda_i = 0$. Let $\mathcal{U}(\mathfrak{g})_0^+ := \mathcal{U}(\mathfrak{g}_0) \cap \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$

Lem 2: $\mathcal{U}(\mathfrak{g})_0^+ \subset \mathcal{U}(\mathfrak{g})_0$ is a 2-sid ideal & $\mathcal{U}(\mathfrak{g})_0 / \mathcal{U}(\mathfrak{g})_0^+ \xrightarrow{\sim} \mathcal{U}(\mathfrak{h})$.

Proof: Assume that $x_1, \dots x_k \in \mathfrak{n}^-$, $x_{k+1}, \dots x_{k+r} \in \mathfrak{h}$, $x_{k+r+1}, \dots x_n \in \mathfrak{n}^+$ $(2k+r=n)$. Note that $\mathcal{U}(\mathfrak{g})_0 = \mathcal{U}(\mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})_0^+$ (decomp'n of vect. spaces). Also note that $\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{g})_0^+$, $\mathcal{U}(\mathfrak{g})_0^+\mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g})_0^+$ and that $\mathcal{U}(\mathfrak{g})_0^+ \mathcal{U}(\mathfrak{g})_0^+ \subset \mathcal{U}(\mathfrak{g})_0^+$. So $\mathcal{U}(\mathfrak{g})_0^+$ is a 2-sided ideal. The decomp'n $\mathcal{U}(\mathfrak{g})_0 = \mathcal{U}(\mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})_0^+$ gives $\mathcal{U}(\mathfrak{h}) = \mathcal{U}(\mathfrak{g})_0 / \mathcal{U}(\mathfrak{g})_0^+$ $\square$

Now note that $Z(\mathcal{U}(\mathfrak{g})) = \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \in \mathcal{U}(\mathfrak{g})_0$. Also note that $\mathcal{U}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$. Restricting $\mathcal{U}(\mathfrak{g})_0 \twoheadrightarrow \mathcal{U}(\mathfrak{h})$ to $Z(\mathcal{U}(\mathfrak{g}))$ we get a required homomorphism. It'll be denoted by φ.

2.5) Verma modules and some homomorphisms

Let $\lambda \in \mathfrak{h}^*$. Define the <u>Verma module</u> $\Delta(\lambda)$ over $\mathcal{U}(\mathfrak{g})$ as

$$\Delta(\lambda) = \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g})\, \mathrm{Span}_{\mathbb{C}}(y, x - \langle \lambda, x \rangle \mid y \in \mathfrak{n}^+, x \in \mathfrak{h})$$

$= \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional rep'n of $\mathfrak{b}$, where $\mathfrak{n}^+$ acts by 0 and $\mathfrak{h}$ acts by λ.

The triangular decomposition of $\mathcal{U}(\mathfrak{g})$ $(= \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+))$ implies $\Delta(\lambda) \underset{\mathcal{U}(\mathfrak{n}_-)}{\simeq} \mathcal{U}(\mathfrak{n}^-)$. A basis in $\Delta(\lambda)$ can be described as follows: Let v_λ denote the image of $1 \in \mathcal{U}(\mathfrak{g})$ in $\Delta(\lambda)$ (a.k.a highest vector). Then the vectors $\overrightarrow{\prod_{\alpha > 0}} f_\alpha^{n_\alpha} v_\lambda$ form a basis in $\Delta(\lambda)$, where the notation is as follows. We write f_α for a nonzero element in $\mathfrak{g}_{-\alpha}$. We order negative roots in some order and take the product in that order. The action of $\mathfrak{h}$ on the basis vector is by the character $\lambda - \sum_{\alpha > 0} n_\alpha \alpha$ (so it's a weight vector)

Ex ($\mathfrak{g} = \mathfrak{sl}_2$) Have basis $v_i = \frac{f^i}{i!} v_\lambda$ $(i \geq 0)$ in $\Delta(\lambda)$ $(\lambda \in \mathbb{C})$

The action of generators is given by:

$f v_i = (i+1) v_{i+1}$, $h v_i = (\lambda - 2i) v_i$, $e v_i = \begin{cases} 0, & i = 0 \\ (\lambda+1-i) v_{i-1}, & i > 0 \end{cases}$

commute e past f^i

In part'r, for $i = n \in \mathbb{Z}_{\geq 0}$ we have $e v_{n+1} = 0$

Now let's describe homomorphisms from $\Delta(\lambda)$ to some other $\mathcal{U}(\mathfrak{g})$-module M. Set $M_\lambda = \{m \in M \mid xm = \langle \lambda, x \rangle m, \forall x \in \mathfrak{h}\}$ and let $M^{\mathfrak{n}_+}$ denote the annihilator of $\mathfrak{n}_+$ in M.

(1) $\mathrm{Hom}(\Delta(\lambda), M) = M_\lambda \cap M^{\mathfrak{n}_+}$

Indeed, ${}^{\mathfrak{g}}\Delta(\lambda)$ is cyclic so any hom is determined by the image of v_λ, the r.h.s. consists of possible images (= ann'r of $\mathcal{U}(\mathfrak{g})\mathrm{Span}_{\mathbb{C}}(y, x - \langle \lambda, x \rangle)$). (♤)

(♤) For example, $\mathrm{End}_{\mathfrak{g}}(\Delta(\lambda)) = \mathbb{C}$

To accomplish step II of the proof of HC isomorphism, we need

Prop 2: Let λ satisfy $\langle \lambda, \alpha_i^\vee \rangle = n \,(\in \mathbb{Z}_{\geq 0})$. Then there is a nonzero homomorphism $\Delta(\lambda - (n+1)\alpha_i) \longrightarrow \Delta(\lambda)$

Proof: By (1), we need to show that $\Delta(\lambda)_{\lambda - (n+1)\alpha_i}$ contains a vector annih'd by $\mathfrak{n}_+$ ($\Leftrightarrow$ by all e_i's). The description of the basis of $\Delta(\lambda)$, we see that $\Delta(\lambda)_{\lambda-(n+1)\alpha_i} = \mathbb{C} \cdot f_i^{n+1} v_\lambda$. We have $e_j f_i^{n+1} v_\lambda = f_i^{n+1} e_j v_\lambda = 0$ and $e_i f_i^{n+1} v_\lambda = 0$ by a computation similar to the example $\square$

2.6) $\mathrm{im}\,\varphi \subset \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$

It'll be convenient for us to write φ_z for $\varphi(z) \in \mathbb{C}[\mathfrak{h}^*]$. We want to show that $\varphi_z(\lambda) = \varphi_z(w \cdot \lambda)$. It's enough to prove this for generators of W, i.e. for simple refl'ns s_i

Lem 3: $z \in Z(\mathcal{U}(\mathfrak{g}))$ acts on $\Delta(\lambda)$ by scalar $\varphi_z(\lambda)$.

Proof: The module $\Delta(\lambda)$ is gen'd by v_λ and since z is central, it's enough to show that $z v_\lambda = \varphi_z(\lambda) v_\lambda$. By the construction of $\varphi(z) \in \mathcal{U}(\mathfrak{h})$, $z - \varphi(z) \in \mathcal{U}(\mathfrak{g})\mathfrak{n}_+$ hence annihilates v_λ. But $\varphi(z) v_\lambda = \varphi_z(\lambda) v_\lambda$ $\square$

Prop'n 3: $\varphi_z(\lambda)=\varphi_z(s_i\cdot\lambda)\ \forall \lambda\in\mathfrak{h}^*$

Clearly, this establishes $\varphi_z\in\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$

Proof: φ_z is a polynomial so it's enough to prove this for all λ in some Zariski dense set e.g. in $\{\lambda \mid \langle\lambda,\alpha_i^\vee\rangle\in\mathbb{Z}_{\geq 0}\}$ So let $\langle\lambda,\alpha_i^\vee\rangle=n$

We have $s_i\cdot\lambda = s_i(\lambda+\rho)-\rho = s_i\lambda+(s_i\rho-\rho) = \lambda-\langle\lambda,\alpha_i^\vee\rangle\alpha_i-\alpha_i$ $=\lambda-(n+1)\alpha_i$. So there is a nonzero homomorphism $\Delta(s_i\cdot\lambda)\to\Delta(\lambda)$

The scalars of the Z-action on these Δ's have to coincide, and by Lem 3 we are done $\square$

2.7) Completion of proof

The isomorphism $\operatorname{gr}\mathcal{U}(\mathfrak{g})=S(\mathfrak{g})$ gives rise to $\operatorname{gr}\mathcal{U}(\mathfrak{g})^G=S(\mathfrak{g})^G$.

It's clear from the construction of $\varphi: \mathcal{U}(\mathfrak{g})^G\to\mathcal{U}(\mathfrak{h})$ that it's compatible w. filtr'ns so it induces a homomorphism of assoc'd graded algebras $\operatorname{gr}\varphi: \operatorname{gr}\mathcal{U}(\mathfrak{g})^G=S(\mathfrak{g})^G\longrightarrow S(\mathfrak{h})=\operatorname{gr}\mathcal{U}(\mathfrak{h})$.

The homomorphism admits a description analogous to φ:

$$\begin{tikzcd} S(\mathfrak{g})^G \arrow[r, hook] \arrow[d, "\operatorname{gr}\varphi"'] & S(\mathfrak{g})_0 \arrow[d] \\ S(\mathfrak{h}) \arrow[r, "\sim"] & S(\mathfrak{g})_0/S(\mathfrak{g})_0^+ \end{tikzcd}$$

To carefully check this is an exercise

But there's also a different description of $\operatorname{gr}\varphi$. Namely, the decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ embeds $\mathfrak{h}^*\hookrightarrow\mathfrak{g}^*$

Exercise: $\operatorname{gr}\varphi(f)=f|_{\mathfrak{h}^*}$ for $f\in S(\mathfrak{g})^G=\mathbb{C}[\mathfrak{g}^*]^G$

Thm 3 (Chevalley): The map $f\mapsto f|_{\mathfrak{h}^*}$ gives an isomorphism $\mathbb{C}[\mathfrak{g}^*]^G\xrightarrow{\sim}\mathbb{C}[\mathfrak{h}^*]^W$ (proof: to be explained in Inv. theory class)

Compl'n of proof of Thm 2: We have $\varphi: \mathcal{U}(\mathfrak{g})^G\longrightarrow\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$

Clearly, $\operatorname{gr}\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}=\mathbb{C}[\mathfrak{h}^*]^W$. As we've explained, $\operatorname{gr}\varphi$: $\mathbb{C}[\mathfrak{g}^*]^G\longrightarrow\mathbb{C}[\mathfrak{h}^*]^W$ is the restriction map. By the Chevalley Thm, $\operatorname{gr}\varphi$ is an iso. Therefore φ is an iso $\square$

3.1) Simple quotient.

Prop 4. Every $\Delta(\lambda)$ has a unique simple quotient.

Proof: This is equiv't to each $\Delta(\lambda)$ has a unique max'l submodule. Note that $\Delta(\lambda)$ decomposes into the sum of wt. space, $\Delta(\lambda) = \bigoplus_\mu \Delta(\lambda)_\mu$. Any submodule R is compatible w. this decomp'n and $R = \Delta(\lambda) \Leftrightarrow R \not\subset \bigoplus_{\mu \neq \lambda} \Delta(\lambda)_\mu$. So the sum of proper submodules is proper and we are done. $\square$

Let $R(\lambda)$ denote the unique max'l submodule of $\Delta(\lambda)$ and $L(\lambda) = \Delta(\lambda)/R(\lambda)$

Exer 1: TFAE:
- $\Delta(\lambda)$ is simple
- $\Delta(\lambda)^{\mathfrak{n}_+} = \mathbb{C} v_\lambda$

To finish this part, let's recall the classification of the finite dimensional $\mathfrak{g}$-modules. By Λ we denote the weight lattice in $\mathfrak{h}^*$, i.e. $\Lambda = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$. By Λ^+ we denote the submonoid of dominant wts: $\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0\}$

Thm 4 (classical) All fin dim'l rep's of $\mathfrak{g}$ are completely reducible. The irreducibles are precisely $L(\lambda)$, $\lambda \in \Lambda^+$ (so they are classified by their highest wts).

3.2) Hom's between Vermas

In the rest of this section we we'll study Hom-spaces between Verma modules. For this we'll need linkage (a.k.a. Bruhat) order $\preceq$ on $\mathfrak{h}^*$.

Def: Let $\lambda, \mu \in \mathfrak{h}^*$. We say that $\lambda \preceq \mu$ if $\exists\ \lambda_0, \dots, \lambda_k \in \mathfrak{h}^*, \beta_1, \dots, \beta_k \in R^+$ s.t.
- $\lambda_0 = \lambda,\ \lambda_k = \mu$
- $\langle \lambda_{i-1} + \rho, \beta_i^\vee \rangle \in \mathbb{Z}_{>0}\ \forall i$
- $\lambda_i = s_{\beta_i} \cdot (\lambda_{i-1})\ \forall i$

Thm 5: The following are true

(i) Any homomorphism between Vermas is injective

(ii) $\dim \operatorname{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \leq 1$

In fact, $\Rightarrow$ also holds but we've not going to prove this

(iii) $\dim \operatorname{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) = 1 \Leftarrow \mu \preceq \lambda$

3.3) Proof of (i)

$\varphi: \Delta(\mu) \to \Delta(\lambda)$; $\Delta(\mu), \Delta(\lambda) \xrightarrow[\mathcal{U}(n_-)]{\sim} \mathcal{U}(n_-)$. So φ is inj've if and only if $a\varphi(v_\mu) \neq 0$ for $a \neq 0, a \in \mathcal{U}(n_-)$. But we can view $\varphi(v_\mu)$ as an el't of $\mathcal{U}(n_-)$. Since this is an algebra w/o zero divisors ((2) of Prop 1), we are done.

3.4) Proof of (ii)

Define a new order on $\mathfrak{h}^*$: $\mu \leq \lambda \overset{def}{\Longleftrightarrow} \lambda - \mu = \sum_{i=1}^r n_i \alpha_i$ w. $n_i \in \mathbb{Z}_{\geq 0}$ $\forall i$.

Exer 2: If $\operatorname{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \neq 0$, then $\mu \leq \lambda$ and $W \cdot \mu = W \cdot \lambda$ (orbits under the ρ-shifted action (note: $\mu \preceq \lambda \Rightarrow \mu \leq \lambda$, $W \cdot \mu = W \cdot \lambda$)

Cor 4: Every Verma module contains a submodule that is an irreducible Verma module

Proof: Pick a minimal (w.r.t. $\leqslant$) μ s.t. $\Delta(\mu) \hookrightarrow \Delta(\lambda)$ (by the 2nd part of Exercise, there are finitely many μ w. $\Delta(\mu) \hookrightarrow \Delta(\lambda)$, so a minimal one indeed exists. By the 1st part, if $\operatorname{Hom}_{\mathfrak{g}}(\Delta(\mu'), \Delta(\mu)) \neq 0$, then $\mu' = \mu$. By the descr'n of $\operatorname{Hom}(\Delta(?), M)$ (eq'n (1)), we have $\Delta(\mu)^{n_+} = \mathbb{C} v_\mu$. By Exer. 1, $\Delta(\mu)$ is irreducible

Prop 5: If $\Delta(\mu)$ is simple, then $\dim \operatorname{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \leq 1$

Proof: Assume the converse, then $\Delta(\mu)^{\oplus 2} \hookrightarrow \Delta(\lambda)$. Let's grade $\Delta(\lambda)$ so that $\deg v_\lambda = 0$ and f_α has degree $\langle \alpha, \rho^\vee \rangle = \sum m_i$, where $\alpha = \sum_i m_i \alpha_i$

We claim that $\dim \Delta(\lambda)_n$ for $n \gg 0$ is a degree $\dim n^+$ polynomial

Indeed, $\dim \Delta(\lambda)_n = \dim S(n^-)_n$ for a similarly defined grading. (*)

Let $F(n)$ be this polynomial. Now define a grading on $\Delta(\mu)$ by putting v_μ in deg $n_0 - \langle \lambda - \mu, \rho^\vee \rangle$ (so that any homomorphism $\Delta(\mu) \hookrightarrow \Delta(\lambda)$ has degree 0). So for $n \gg 0$, we have $\dim \Delta(\mu)_n = F(n - n_0)$. The inclusion $\Delta(\mu)^{\oplus 2} \hookrightarrow \Delta(\lambda)$ implies $2 \dim \Delta(\mu)_n = 2F(n - n_0) \leq F(n) = \dim \Delta(\lambda)_n$. The middle inequality is nonsense $\square$

Proof of (ii) in gen'l: Let $\varphi_1, \varphi_2: \Delta(\mu) \to \Delta(\lambda)$ be linearly indep't

(*) And now our claim follows from a standard result in commutative algebra homomorphisms. Let $\Delta(\mu')$ be an irreducible Verma submodule of $\Delta(\mu)$. By Prop 5, $\varphi_1|_{\Delta(\mu')}, \varphi_2|_{\Delta(\mu')}$ are proportional, say $\varphi_2|_{\Delta(\mu')} = a\,\varphi_1|_{\Delta(\mu')}$. But then $\varphi_2 - a\varphi_1$ is a nonzero hom'm vanishing on $\Delta(\mu')$. By (i), this is impossible $\square$

3.5) Proof of (iii) - that's a fun part!

First, we deal with the case $\mu\in\Lambda$ (of course, $\lambda\in\Lambda$). We need to show that for $\alpha\in R_+$ w. $\langle\mu+\rho,\alpha^\vee\rangle<0$, we have $\Delta(\mu)\hookrightarrow\Delta(s_\alpha\cdot\mu)$. For $\mu+\rho\in\Lambda^+$, the claim is vacuous. So suppose that we know our claim for all $\mu'\succ\mu$. Pick $\alpha_i\in S$ w. $\mu\prec s_i\cdot\mu$ (which exists if $\mu+\rho\notin\Lambda^+$). Set $\lambda = s_\alpha\cdot\mu$, $\tilde{\mu}=s_i\cdot\mu$, $\tilde{\lambda}=s_i\cdot\lambda$, $\tilde{\alpha}=s_i(\alpha)$. If $\alpha\neq\alpha_i$, then $\tilde{\mu}\prec\tilde{\lambda}$: $\tilde{\lambda}=s_{\tilde{\alpha}}\cdot\tilde{\mu}$ and $\lambda-\mu>0\Rightarrow\tilde{\lambda}-\tilde{\mu}>0$. By induction, $\Delta(\tilde{\mu})\subset\Delta(\tilde{\lambda})$. Also by the proof of Prop 3, $\Delta(\mu)\subset\Delta(\tilde{\mu})$. If $\tilde{\lambda}\preceq\lambda$, then $\Delta(\tilde{\lambda})\subset\Delta(\lambda)$ and we are done. So we only need to consider the case $\lambda\prec\tilde{\lambda}$, where $\Delta(\lambda)\subset\Delta(\tilde{\lambda})$. Note that

(1) $v_\lambda\in\mathbb{C}f_i^a v_{\tilde{\lambda}}$, $v_\mu\in\mathbb{C}f_i^b v_{\tilde{\mu}}$ for some $a,b\in\mathbb{Z}_{>0}$

(2) $v_\mu\in\mathbb{C}e_i^c f_i^c v_\mu$ $\forall c\in\mathbb{Z}_{>0}$ - this follows from a computation similar to the $\mathfrak{sl}_2$-example b/c $\langle\mu+\rho,\alpha_i^\vee\rangle<0$

We need to show $v_\mu\in\mathcal{U}(\mathfrak{n}_-)f_i^a v_{\tilde{\lambda}}$. Thanks to (1) & (2), enough to show that $f_i^c v_{\tilde{\mu}}\in\mathcal{U}(\mathfrak{n}_-)f_i^a v_{\tilde{\lambda}}$ for some $c>b$ i.e. that $f_i^c x\in\mathcal{U}(\mathfrak{n}_-)f_i^a$ for $x\in\mathcal{U}(\mathfrak{n}_-)$ s.t $x v_{\tilde{\lambda}}=v_{\tilde{\mu}}$. But, for $c\gg 0$, this follows from the observation to $[f_i,\cdot]:\mathcal{U}(\mathfrak{n}_-)\circlearrowleft$ is a locally nilpotent operator. This settles the case $\mu\in\Lambda$

Let's consider the general case. Let V denote affine subspace of $\mathfrak{h}^*$ consisting of all λ s.t $\langle\lambda+\rho,\alpha^\vee\rangle=m$, where $m\in\mathbb{Z}_{>0}$ is fixed. We need to check that there is an element in $\Delta(\lambda)_{\lambda-m\alpha}$ annihilated by all e_i's. Let $e=\bigoplus_i e_i:\Delta(\lambda)_{\lambda-m\alpha}\longrightarrow\bigoplus_i\Delta(\lambda)_{\lambda-m\alpha+\alpha_i}$. Note that the spaces $U_1=\Delta(\lambda)_{\lambda-m\alpha}$, $U_2=\bigoplus_i\Delta(\lambda)_{\lambda-m\alpha+\alpha_i}$ are independent of λ, but $e=e(\lambda)$ depends on λ. We need to show that $\ker e(\lambda)\neq 0$

But $e(\lambda)$ depends on λ polynomially so the condition $\ker e(\lambda) \neq 0$ defines a Zariski closed condition on λ. The intersection $\Lambda \cap V$ is Zariski dense and we already know that $\ker e(\lambda) \neq 0$ $\forall \lambda \in \Lambda \cap V$. This finishes the proof (iii)

4*) Verma modules and their simple quotients belong to category $\mathcal{O}$ (to be defined by Daniil in his talk). For such modules it makes sense to speak about characters. The characters of Vermas are known and easy to write down. A fundamental question is to compute characters of simples.

For finite dimensional modules the answer is classical, it's given by the Weyl character formulas. In general, the answer was only found in the 80's. First (1979) Kazhdan and Lusztig conjectured a formula that expresses characters of simples via characters of Vermas via the so called Kazhdan-Lusztig (a.k.a. canonical) bases in the Hecke algebras. The conjecture was soon proved by Beilinson-Bernstein and by Brylinski-Kashiwara. A logic of the proof is as follows:

cat'y $\mathcal{O} \longleftrightarrow \mathcal{D}$-modules on flag var'y $G/B \longleftrightarrow$ perverse sheaves on G/B

$\rightarrow$ proof using a pretty advanced machinery.

The first step (based on the Beilinson-Bernstein localization Thm) is relatively easy, while the other two steps are hard.

In the 90's Soergel found an alternative way to pass from $\mathcal{O}$ to perverse sheaves: via Soergel modules. These are certain graded modules over $\mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]^W_+)$. More precisely, he proved that the category of projective objects in $\mathcal{O}$ (that will appear in Daniil's talk) is equivalent to category of Soergel modules. He then related the latter to the perverse sheaves on G/B (that allows to compute the characters of projectives - info equiv't to characters of simples). Still the complicated & non-elementary machinery is essential for his approach