SOME HECKE ALGEBRAS ASSOCIATED TO THE P-ADIC GROUP $GL(V)$

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1. Introduction

We focus on the special case $G = GL(V)$, where V is a vector space of dimension n over a p-adic field k. But first, we recall a number of statements from the previous talk, with the example $G = GL(V)$ in mind.

Let G be a locally compact, totally disconnected, Hausdorff topological group with a neighborhood basis $\{K_i\}_i$ of the identity consisting of compact, open, normal subgroups. Often $K \subset G$ will denote a compact, open subgroup of G. We consider representations (ρ, V) of G, where V is often infinite-dimensional. We say the representation is *smooth* if $V = \cup_K V^K$, i.e., if every $v \in V$ is fixed by some compact, open subgroup of G. We say the representation is *admissible* if $\dim_{\mathbb{C}}(V^K) < \infty$, for all open, compact subgroups K.

We consider the set $C_c^\infty(G) := \{f : G \to \mathbb{C} : f \text{ is locally constant and compactly supported}\}$. It is an associative algebra over $\mathbb{C}$ under convolution. The important property of this algebra (sometimes denoted the Hecke algebra $\mathcal{H}(G)$) is that there is an equivalence of categories between the smooth representations of G and the representations of $\mathcal{H}(G)$.

Once we recall how to obtain $(\widetilde{\rho}, V)$ from (ρ, V), we explain the connection between representations of $\mathcal{H}(G)$ generated by χ_K-fixed vectors and representations of $\mathcal{H}(G//K)$. We study the structures of two particular Hecke algebras $\mathcal{H}(G//K), \mathcal{H}(G//J)$, where $G = GL(V)$, $K \subset G$ is a maximal compact, open subgroup and $J \subset G$ is the *Iwahori subgroup* of G.

2. Reminders about representations on $\mathcal{H}(G)$

In this section, we consider the general scenario given in the intro. It admits unique, up to scalars, left and right Haar measures. Any reductive p-adic group is *unimodular*, meaning that they coincide. We call this measure μ.

Lemma 2.1. *Let $f \in \mathcal{H}(G)$, then there is a compact open subgroup $K < G$ such that f is right K-invariant.*

Proof. There is a neighborhood basis $\{xK_i\}$ of open, compact sets around $x \in G$. Since f is locally constant, there is some xK_x on which f is constant. Since f is compactly supported, there is a compact $C \subseteq G$ on which f is supported. For being compact, C is covered by finitely many open sets xK_x, let us say $C \subseteq \bigcup_i x_i K_{x_i}$. The set $K = \bigcap_i K_{x_i}$ is clearly an open, compact subgroup of G. One can check f is right K-invariant. $\square$

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From the lemma, one can write an integration of $f \in \mathcal{H}(G//K)$ as a finite sum:

$$\int_G f(g)dg = \sum_{x \in G/K} f(x)\mu(K) \tag{2.1}$$

where K is as in Lemma 2.1. The functor from the smooth representations of G to representations of $\mathcal{H}(G)$ is $(\rho, V) \mapsto (\widetilde{\rho}, V)$, given by

$$\widetilde{\rho}(f) \cdot v := \int_G f(g)\rho(g)vdg \text{ for all } f \in \mathcal{H}(G) \text{ and } v \in V. \tag{2.2}$$

This functor induces an equivalence between smooth representations of G and representations of $\mathcal{H}(G)$. Moreover one can add some restrictions to both sides:

Proposition 2.2. *Let (ρ, V) be a smooth representation of G and $(\widetilde{\rho}, V)$ be the induced representation of $\mathcal{H}(G)$. Then the following statements hold.*

1. *$W \subset V$ is a subrepresentation of G if and only if W is $\widetilde{\rho}(f)$-invariant for all $f \in \mathcal{H}(G)$.*
2. *(ρ, V) is admissible if and only if $\widetilde{\rho}(f)$ has finite rank for all $f \in \mathcal{H}(G)$.*
3. *The representation (ρ, V) is generated by its fixed K-vectors if and only if $(\widetilde{\rho}, V)$ is generated by its fixed χ_K-fixed vectors.*

The Hecke algebra

$$\mathcal{H}(G//K) = \{f \in \mathcal{H}(G) : f(kgk') = f(g) \text{ for all } g \in G; k, k' \in K\} \tag{2.3}$$

is essential to the study of representations generated by χ_K-fixed vectors. The element χ_K is zero outside of K, constant on K and such that $\int_G \chi_K dg = 1$. It satisfies

1. $\chi_K * \chi_K = \chi_K$.
2. For all $f \in \mathcal{H}(G)$, we have $\chi_K * f = f$ if and only if $f(kg) = f(g)$, for all $k \in K$.
3. For all $f \in \mathcal{H}(G)$, we have $f * \chi_K = f$ if and only if $f(gk) = f(g)$, for all $k \in K$.

From (2) and (3) above, it follows that

$$\mathcal{H}(G//K) = \chi_K * \mathcal{H}(G) * \chi_K \tag{2.4}$$

How do we relate irreducible representations of $\mathcal{H}(G)$ generated by χ_K fixed vector with those irreducible representations of $\mathcal{H}(G//K)$?

This relation holds in a more general case. If A is an associative algebra over $\mathbb{C}$, and $e \in A$ is an idempotent, then eAe is a subalgebra of A. (think of $A = \mathcal{H}(G)$, $e = \chi_K$ and $eAe = \mathcal{H}(G//K)$). Let $\mathcal{M}(A)$ be the category of representations of A. Then there are natural induction and restriction functors $r : \mathcal{M}(A) \to \mathcal{M}(eAe)$, $Y \mapsto eY$, and $i : \mathcal{M}(eAe) \to \mathcal{M}(A)$, $Z \mapsto Ae \otimes_{eAe} Z$.

Furthermore, if we let $\widehat{A}$ be the irreducible representations of A and $\mathcal{M}(A, e) = \{V \in \mathcal{M}(A) : V = AeV\}$ be the A-modules generated by e-fixed vectors, then under certain hypotheses (that holds in our case $A = \mathcal{H}(G)$), we have that r restricts to a bijection $r : \widehat{A} \cap \mathcal{M}(A, e) \xrightarrow{\sim} \widehat{eAe}$.

Thus, we see that to understand irreducible representations of G generated by K-fixed vectors, one could study the irreducible representations of $\mathcal{H}(G//K)$. In what is left, we study the structures of some of these Hecke algebras.

3. Preliminaries in the structure of $G = GL(V)$

3.1. **Lattice flags.** Let $k \supseteq \mathbb{Q}_p$ be a p-adic field, with p-adic norm $||\cdot||_p$. The ring of integers is $\mathcal{O}$ is the integral closure of $\mathbb{Z}_p$ in k. One has that $\mathcal{O} = \{a \in k : ||a||_p \leq 1\}$ and $\mathcal{O}$ is an open, compact subgroup of k. The ring of integers $\mathcal{O}$ is a DVR. Let $\mathfrak{m}$ be its maximal ideal and π be a uniformizing parameter. We call $\overline{k} = \mathcal{O}/\mathfrak{m}$ the residue field of k, which is a finite field (if $k = \mathbb{Q}_p^n$, $\mathcal{O} = \mathbb{Z}_p^n$, then $\overline{k} = \mathbb{Z}_p^n/(p\mathbb{Z}_p)^n \cong \mathbb{F}_{p^n}$). We let q be the size of $\overline{k}$. Let V be a vector space of dimension n over k. Then V is given a topology after identifying it with k^n, space which has the product topology $||\cdot||_p^n$. This topology does not depend on the choice of basis.

Definition 3.1. A lattice $\Lambda \subset V$ is a compact, open $\mathcal{O}$-module.

Proposition 3.2. *Any lattice $\Lambda \subset V$ is isomorphic to $\mathcal{O}^n$ as an $\mathcal{O}$-module.*

We can say even more. If Λ is a lattice in V, then $\pi\Lambda$ is also a lattice, it is contained in Λ, and $\overline{\Lambda} = \Lambda/\pi\Lambda$ is a $\overline{k} = \mathcal{O}/\mathfrak{m}$-vector space. Assume that $e_1, \ldots, e_n \in \Lambda$ are such that their images $\overline{e}_1, \ldots, \overline{e}_n$ in $\overline{\Lambda} = \Lambda/\pi\Lambda$ form a $\overline{k}$-basis of $\overline{\Lambda}$. From Nakayama's lemma, $e_1, \ldots, e_n$ spans Λ as an $\mathcal{O}$-module. From Proposition 3.2, $e_1, \ldots, e_n$ is an $\mathcal{O}$-basis of Λ. Now if $v \in V$ is arbitrary, there exists $N \in \mathbb{N}$ large enough such that $\pi^N v \in \Lambda$. Then $\pi^N v = \sum_{j=1}^n \beta_j e_j$, for some $\beta_j \in \mathcal{O}$. This implies $v = \sum_{j=1}^n (\pi^{-N}\beta_j)e_j$, with each $\pi^{-N}\beta_j \in k$. Thus $E = \{e_j\}_{j=1,\ldots,n}$ spans V as a k-vector space. It follows that E is also a k-basis of V. We summarize our discussion as

Proposition 3.3. *Let $\Lambda \subset V$ be a lattice, $E = \{e_j\}_{j=1,2,\ldots,m} \subset \Lambda$ and $\overline{E} = \{\overline{e}_j\}_{j=1,2,\ldots,m} \subset \overline{\Lambda} = \Lambda/\pi\Lambda$ be the set of images of e_j in $\overline{\Lambda}$. If $\overline{E}$ is a $\overline{k}$-vector space for $\overline{\Lambda}$, then E is an $\mathcal{O}$-basis for Λ and a k-basis for V.*

The Iwahori-Bruhat decomposition (for $G = GL(V)$) to be proved later needs the definition of "lattice flags" $\mathcal{L}$, which will play the role of flags of subspaces.

Definition 3.4. A set $\mathcal{L}$ of lattices is a *lattice flag* if
(a) it is totally ordered by inclusion, and
(b) it is invariant under multiplication by $k^\times$.

Condition (b) can be reformulated. Let $\mathcal{L}$ be a lattice flag and $\Lambda_0 \in \mathcal{L}$. Let $x = \pi^n u \in k^\times$, where $u \in \mathcal{O}^\times$. Then $x\Lambda_0 = \pi^n(u\Lambda_0) = \pi^n\Lambda_0$, where the latter equality holds because Λ_0 is an $\mathcal{O}$-module. Therefore (b) holds if and only if (b') $\pi^{\pm 1}\Lambda_0 \in \mathcal{L}$ whenever $\Lambda_0 \in \mathcal{L}$.

3.2. **Stabilizers of lattices.** For a lattice $\Lambda \subset V$, we let $K(\Lambda)$ be the subgroup of $GL(V)$ consisting of automorphisms of Λ, i..e,

$$K(\Lambda) = \{g \in GL(V) : g\Lambda = \Lambda\} \tag{3.1}$$

Proposition 3.5. *There is a unique conjugacy class of maximal compact subgroups of $GL(V)$, consisting of the stabilizers $K(\Lambda)$ of lattices Λ.*

Proof. Choose a basis $E = \{e_j\}_{j=1,\ldots,n} \subset \Lambda$, as in 3.3. In this basis, $G = GL_n(k)$ and $K(\Lambda) = GL_n(\mathcal{O})$. It is not difficult to see that $GL_n(\mathcal{O})$ is an open, compact subset of $GL_n(k)$. If Λ' is another lattice, we can find $g \in GL(V)$ such that $g(\Lambda) = \Lambda'$. (For example, by choosing E, resp. E', to be $\mathcal{O}$-bases of Λ, resp. Λ', and k-bases of V, and g be the matrix of change of basis from E to E'.) It follows that $K(\Lambda') = gK(\Lambda)g^{-1}$, so $K(\Lambda)$ and $K(\Lambda')$ are conjugate.

Let H be any compact subgroup of $GL(V)$. Since $K(\Lambda)$ is open, $H \cap K(\Lambda)$ has finite index in H. Then the lattices $\{h(\Lambda) : h \in H\}$ form a finite set. Therefore the sum of such lattices $\overline{\Lambda}$ is again a lattice in V, and is clearly stabilized by H. Hence $H \subset K(\overline{\Lambda})$, thus implying that any maximal compact subgroup of $GL(V)$ is a stabilizer of a lattice. $\square$

Corollary 3.6. *If K is a maximal compact, open subgroup of $G = GL(V)$, then there exists a basis of V such that $G = GL_n(k)$ and $K = GL_n(\mathcal{O})$.*

Proof. From Proposition 3.5, there is a lattice Λ such that $K = K(\Lambda)$. From proposition 3.3, there is a set $E = \{e_1, \ldots, e_n\}$ which is an $\mathcal{O}$-basis of Λ and a k-basis of V. In terms of this basis, we have $V = k^n$ and $\Lambda = \mathcal{O}^n$. Therefore we have $G = GL_n(k)$ and $K = GL_n(\mathcal{O})$. $\square$

4. Iwahori-Bruhat decomposition and structure of $\mathcal{H}(GL(V)//K)$

Theorem 4.1. (Bruhat Decomposition) *Let G be a reductive group, B a Borel subgroup and W its Weyl group. Then $G = BWB$, or more precisely,*

$$G = \coprod_{w \in W} BwB.$$

We give an equivalent statement: the *Geometric Bruhat Decomposition*. Both have analogues in the p-adic case, where lattice flags replace flags. We consider *line decompositions* $V = \bigoplus_j L_j$, where each L_j is a 1-dimensional subspace of V. A line decomposition is said to be compatible with a flag $\mathcal{F} = \{0 = U_0 \subset U_1 \subset \ldots \subset U_k = V\}$ if $U_j = \bigoplus_j (L_j \oplus U_i)$ for all $i > 0$.

Proposition 4.2. *$GL(V) = BWB$ if and only if for any two flags $\mathcal{F}_1$ and $\mathcal{F}_2$, there exists a line decomposition of V compatible with both $\mathcal{F}_1$ and $\mathcal{F}_2$.*

Proof. ($\Longrightarrow$) Let $\mathcal{F}_1, \mathcal{F}_2$ be two flags, that we can assume are maximal, and let $B = Stab_{GL(V)}\mathcal{F}_1$ and $W = Stab_{GL(V)}F_1$, where F_1 is a basis of V, compatible with $\mathcal{F}_1$. Let $g \in GL(V)$ be such that $g(\mathcal{F}_1) = \mathcal{F}_2$. Since $G = BWB$, we write $g = b_1wb_2$. We claim that $E = b_1(F_1)$ is a basis of V compatible with both $\mathcal{F}_1$ and $\mathcal{F}_2$. Since $b_1 \in Stab_{GL(V)}\mathcal{F}_1$, then E is compatible with $\mathcal{F}_1$. We consider $b_1wb_1^{-1}(E) = b_1w(F_1)$, which is just a reordering of the elements of E. But it also equals $gb_2^{-1}(F_1)$ and since $b_2^{-1} \in Stab_{GL(V)}\mathcal{F}_1$, we have that $b_2^{-1}(F_1)$ is a basis of V, compatible with $\mathcal{F}_1$, so $gb_2^{-1}(F_1)$ is a basis of V compatible with $\mathcal{F}_2$.

($\Longleftarrow$) Let $g \in GL(V)$ be arbitrary and let $\mathcal{F}_1$ be a complete flag such that $B = Stab_{GL(V)}\mathcal{F}_1$. Also let F_1 be a compatible basis for $\mathcal{F}_1$, such that $W = Stab_{GL(V)}F_1$. Set $\mathcal{F}_2 := g\mathcal{F}_1$ and $F_2 := g(F_1)$ be a compatible basis for $\mathcal{F}_2$. By assumption, there is a compatible basis E for both $\mathcal{F}_1$ and $\mathcal{F}_2$. Now choose $b_1 \in B$ such that $b_1(F_1) = E$. Since E is compatible with $\mathcal{F}_1$ and $\mathcal{F}_2$, we have that $F_1 = b_1^{-1}(E)$ is compatible with both $b_1^{-1}\mathcal{F}_1 = \mathcal{F}_1$ and $b_1^{-1}\mathcal{F}_2 = b_1^{-1}g(\mathcal{F}_2) = \mathcal{F}_3$. As such F_1 exists, then there is a permutation $w \in W$ such that $w\mathcal{F}_1 = \mathcal{F}_3$, and it follows that $w^{-1}b_1^{-1}g = b_2$ belongs to $Stab_{GL(V)}\mathcal{F}_1 = B$. It follows that $g = b_2wb_1 \in BwB$. Hence $GL(V) = BWB$. $\square$

Corollary 4.3. (Geometric Bruhat Decomposition) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are any two flags of V, then there is a line decomposition that is compactible with both $\mathcal{F}_1$ and $\mathcal{F}_2$.*

The lattice-analogue of the geometric Bruhat decomposition is:

Theorem 4.4. (Geometric Iwahori-Bruhat decomposition) *If $\mathcal{L}$ and $\mathcal{M}$ are any two lattice flags, then there is a line decomposition $V = \oplus_i L_i$ compatible with both $\mathcal{L}$ and $\mathcal{M}$.*

Before sketching the proof of this theorem, we make the relation between lattice flags and flags of subspaces more explicit.

Let $\mathcal{L}$ be any lattice flag and $\Lambda_0 \in \mathcal{L}$ be arbitrary. If $\Lambda' \in \mathcal{L}$ is any other element of $\mathcal{L}$, then $\pi^m \Lambda' \subset \Lambda_0$ for sufficiently large m. If we choose the smallest $m \in \mathbb{Z}$ for which this holds, then $\pi^{m-1}\Lambda'$ does not belong to Λ_0. As $\mathcal{L}$ is totally ordered by inclusion, then $\Lambda_0 \subset \pi^{m-1}\Lambda' \Longrightarrow \pi\Lambda_0 \subset \pi^m\Lambda' \subset \Lambda_0$. Thus reduction modulo $\pi\Lambda_0$ attaches to $\pi^m\Lambda'$ the subspace $U_{\Lambda'} \subset \overline{\Lambda}_0 = \Lambda_0/\pi\Lambda_0$ of the $\overline{k}$-vector space $\overline{\Lambda}_0$. It is clear that $\pi^m\Lambda'$ can be recovered from $U_{\Lambda'}$ as the unique lattice containing $\pi\Lambda_0$ and reducing to $U_{\Lambda'}$ modulo $\pi\Lambda_0$. If we have $\pi^m\Lambda'$, all multiples of Λ' can also be recovered. Assume Λ'' is any other lattice of $\mathcal{L}$ and $\pi\Lambda_0 \subset \pi^p\Lambda'' \subset \Lambda_0$. If $\pi^m\Lambda' \subset \pi^p\Lambda''$, then it is not difficult to see that it corresponds to inclusions of subspaces $U_{\Lambda'} \subset U_{\Lambda''}$ of $\overline{\Lambda}_0$. So the lattice flag $\mathcal{L}$ determines and is determined by a flag $\overline{\Lambda}_0$. Conversely, given a lattice Λ_0 and a flag $\{U_i\}$ in $\overline{\Lambda}_0 = \Lambda_0/\pi\Lambda_0$, we can form lattices Λ_i such that $\pi\Lambda_0 \subset \Lambda_i \subset \Lambda_0$ and $\Lambda_i/\pi\Lambda_0 = U_i$. Then taking all multiples of $\pi^m\Lambda_i$ of these lattices, it is easy to see that we obtain a lattice flag. Thus we conclude the following

Proposition 4.5. *All lattice flags containing a given lattice Λ_0 are in bijection with all flags of subspaces in the $\overline{k}$-vector space $\overline{\Lambda}_0$.*

From the proposition, it follows that any lattice flag can be extended into a maximal one. Also, in a maximal lattice flag, the quotient of consecutive lattices Λ'/Λ'' is 1-dimensional/$\overline{k}$.

Proof. (Sketch) Assume $\mathcal{L}$ and $\mathcal{M}$ are maximal flags. Select any $\Lambda_0 \in \mathcal{L}$. From above, $\mathcal{L}$ is associated to a flag $\overline{\mathcal{F}}(\mathcal{L})$ in $\overline{\Lambda}_0 = \Lambda_0/\pi\Lambda_0$. Now construct other flag in Λ_0 as follows. Forr each $M \in \mathcal{M}$, set $\widetilde{M} = (M \cap \Lambda_0) + \pi\Lambda_0$, a lattice between Λ_0 and $\pi\Lambda_0$.

So each M defines a subspace $U(M)$ of $\overline{\Lambda}_0$. For small M, $U(M) = 0$, while for large M, $U(M) = \overline{\Lambda}_0$. Successive quotients are 1-dimensional over $\overline{k}$, so the subspaces $\{U(M)\}$ define a maximal flag $\overline{\mathcal{G}}(\mathcal{M})$ in $\overline{\Lambda}_0$.

By the geometric Bruhat decomposition in $GL(\overline{\Lambda}_0)$, we can find a basis $\{\overline{z}_j\}$ compatible with both $\overline{\mathcal{F}}(\mathcal{L})$ and $\overline{\mathcal{G}}(\mathcal{M})$. $\mathcal{F}$ is defined by lattices between Λ_0 and $\pi\Lambda_0$, so any lifts $\{z_j\}$ make a line decomposition of V compatible with $\mathcal{L}$.

Also, $\overline{z}_j$ span $U(M_2)/U(M_1)$ for successive quotients $M_1 \subset M_2$. So M_1 is the largest subspace for which $\overline{z}_j \notin U(M_1)$, and M_2 is the smallest subspace for which $\overline{z}_j \in U(M_2)$. Thus we may lift $\overline{z}_j$ to some $z_j \in M_2$. The claim is that the $\{z_j\}$ make the desired line decomposition. Checking this is an exercise. □

Remark 4.6. There is an easier way to prove this theorem, by proving first the Cartan decomposition of $GL(V)$ (see below) via Gauss elimination. The advantages of proof above is that it is coordinate-free and that illustrates the relation between lattice flags and flags of subspaces.

The geometric version of the Iwahori-Bruhat decomposition also has a version where $GL(V)$ is decomposed. The Borel subgroup B is replaced by the stabilizer $J = J(\mathcal{L})$ of the maximal lattice flag $\mathcal{L}$. If $V = \oplus_j L_j$ is a line decomposition of V that is compatible with $\mathcal{L}$, then let A

be the group of transformations which stabilize all the lines and let $\widetilde{W} = AW$ be the "affine Weyl group" of transformations which stabilize the collection $\{L_j\}_j$, then

$$GL(V) = J(\mathcal{L})\widetilde{W}J(\mathcal{L}). \tag{4.1}$$

From the Iwahori-Bruhat decomposition, the following *Cartan decomposition*

$$GL(V) = K(\Lambda_0)AK(\Lambda_0) = KAK, \tag{4.2}$$

where Λ_0 is a lattice of $\mathcal{L}$, is seen to be true. Under a suitable choice of basis for V, we have $GL(V) = GL_n(k)$, $K(\Lambda_0) = GL_n(\mathcal{O})$, A is the subgroup of diagonal matrices in $GL_n(k)$ and J is the subgroup of matrices in

$$\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} \\ \mathfrak{m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ \mathfrak{m} & \cdots & \mathfrak{m} & \mathcal{O}^\times \end{pmatrix}.$$

Theorem 4.7. *If K is a maximal open, compact subgroup of $GL(V)$, then $\mathcal{H}(GL(V)//K)$ is commutative.*

Proof. We use an elementary technique of Gelfand: find an antiautomorphism of $\mathcal{H}(GL(V)//K)$ that is the identity. First, fix a basis of V for which $GL(V) = GL_n(k)$, $K = GL_n(\mathcal{O})$ and A are the diagonal matrices in $GL_n(k)$.

In this basis, the transpose map $^t : GL(V) \to GL(V)$ is an antiautomorphism that fixes K and A. But since $GL(V) = KAK$ by the Cartan decomposition, then the transpose induces an antiautomorphism of $\mathcal{H}(GL(V)//K)$, via $f \mapsto \widetilde{f} : \widetilde{f}(g) = f(g^t)$, it is the identity on $GL(V)$. □

Remark 4.8. It holds that $\mathcal{H}(G//K) \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$, from which commutativity is obvious. This does not follow from this proof, but from a more refined decomposition of $GL(V)$.

5. Structure of $\mathcal{H}(GL(V)//J)$

5.1. **The extended affine Weyl group.** For $G = GL(V)$, the Weyl group W is the group of permutations S_n, generated by transpositions $s_1, \dots, s_{n-1}$, where s_i is the identity matrix with i and $i+1$ row switched. The extended affine Weyl group $\widetilde{W}^\circ$ adds two additional generators s_0 and t, where

$$s_0 = \begin{pmatrix} 0 & & & & \pi^{-1} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \pi & & & & 0 \end{pmatrix}$$

$$t = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ \pi & & & & 0 \end{pmatrix}.$$

It is a group that contains all diagonal matrices whose entries are powers of π. The choice of t is so that it normalizes the Iwahori subgroup J. One can then verify that $\widetilde{W}^\circ$ is the group presented as $< s_0, s_1, \ldots, s_{n-1}, t | R >$, where R is the set of relations

$$\begin{aligned} s_i^2 &= 1 \text{ for all } 0 \leq i \leq n-1 \\ (s_i s_j)^{m_{ij}} &= 1 \text{ where } m_{i,i+1} = 3 \text{ and } m_{ij} = 2 \text{ whenever } |i-j| \pmod n > 1 \\ t s_j t^{-1} &= s_{j-1} \text{ for all } 1 \leq j \leq n-1 \end{aligned}$$

We define the *length function on* $\widetilde{W}^\circ$ as the map $l : \widetilde{W}^\circ \longrightarrow \mathbb{N}$ that sends each $w \in \widetilde{W}^\circ$ to the minimum number of s_j appearing in some expression of w.

Exercise: Verify that the Haar measure on (the unimodular group) $GL_n(k)$ can be normalized so that $\mu(JwJ) = q^{l(w)}$, for all $w \in \widetilde{W}^\circ$.

5.2. **Iwahori-Bruhat presentation.** We now consider the basis $f_g = \chi_{JgJ}$, $g \in J\backslash G/J$, of $\mathcal{H}(G//J)$. The following lemma works for any compact, open subgroup of G, replacing J.

Lemma 5.1. *If $f_x * f_y = \sum_z a_{xy}^z f_z$, then $a_{xy}^z \in \mathbb{Z}$, and*

$$\mu(JxJ)\mu(JyJ) = \sum_z a_{xy}^z \mu(JzJ).$$

Proof. As J is compact, and $g^{-1}Jg \cap J$ is an open subgroup of J, then $J/(g^{-1}Jg \cap J)$ is finite. Write $JgJ = \bigcup_{i=1}^m k_i gJ$, for $k_i \in J/(g^{-1}Jg \cap J)$. Then

$$\begin{aligned} f_x = \chi_{JxJ} &= \sum_i \chi_x J = \sum_i \delta_{k_i x} * \chi_J \\ f_y &= \sum_j \delta_{\widetilde{k}_j x} * \chi_J \end{aligned}$$

Using that f_y is left J-invariant (so that $\chi_J * f_y = f_y$), we have

$$f_x * f_y = \sum_{i,j} \delta_{k_1, x} \delta_{\widetilde{k}_j, y} * \chi_J$$

from which the first statement follows. The second statement follows from the first by integrating over G using the Haar measure μ. □

Corollary 5.2. *If $\mu(JxJ)\mu(JyJ) = \mu(JxyJ)$, then $f_x * f_y = f_{xy}$.*

From the normalization $\mu(JwJ) = q^{l(w)}$ and Corollary 5.2, it follows that $f_x * f_y = f_{xy}$, whenever $l(x) + l(y) = l(xy)$. There is one additional constraint $f_{s_i}^2 = (q-1) f_{s_i} + q f_1$, whose verification is left as an exercise. These are all relations, as asserted by

Theorem 5.3. *$\mathcal{H}(G//J)$ is the algebra generated by f_{s_i}, $0 \leq i < n$, and f_t subject to*

1. $f_{s_i} * f_{s_i} = (q-1) f_{s_i} + q f_1$.
2. $f_{s_i} * f_{s_j} * f_{s_i} * ** = f_{s_j} * f_{s_i} * f_{s_j} * **$, *for any* i, j.
3. $f_t f_{s_i} = f_{s_{i+1}} f_t$, *for any* $0 \leq i < n$.

This presentation shows that the structure of $\mathcal{H}(GL(V)//J)$ is similar to that of a Coxeter group, and allows us to see it as a deformation of $\widetilde{W}^\circ$. However, it obscures the abelian subgroup generated by f_g, where g runs over the diagonal matrices within $\widetilde{W}^\circ$. This is best seen if one uses the Bernstein-Zelevinski presentation of $\mathcal{H}(GL(V)//J)$.

5.3. **Bernstein-Zelevinski presentation.** In the affine Weyl group $\widetilde{W}^\circ$, we set

$$a_k = \begin{pmatrix} \pi^{-1} & & & & \\ & \ddots & & & \\ & & \pi^{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where the first k entries along the diagonal are π^{-1}. They generate a free semigroup of rank n inside $\widetilde{W}^\circ$. It is easy to check that $l(a_k) = l(n-k)$. One can verify easily that $a_k s_k a_k s_k = a_{k-1} a_{k+1}$.

This implies $a_k s_k a_k = a_{k-1}a_{k+1}s_k$. Both of these words are reduced and $l(a_k s_k) + l(a_k) = l(a_{k-1}) + l(a_{k+1}) + l(s_k)$. Therefore

$$f_{a_k s_k} f_{a_k} = f_{a_{k-1}} f_{a_{k+1}} f_{s_k} \tag{5.1}$$

is valid in $\mathcal{H}(G//J)$. Also, one has $l(a_k s_k) = l(a_k) - 1$, which implies

$$f_{a_k} = f_{a_k s_k} f_{s_k}. \tag{5.2}$$

If we set

$$\begin{aligned} T_k &= q^{-1/2} f_{s_k} \\ y_k &= q^{-(n-2k+1)/2} f_{a_k} f_{k-1}^{-1}, \end{aligned}$$

then equations 5.1 and 5.2 yield the following Bernstein-Zelevinski presentation of $\mathcal{H}(GL(V)//J)$:

$$\begin{aligned} T_k y_k - s_k(y_k) T_k &= (q^{1/2} - q^{-1/2}) \frac{s_k(y_k) - y_k}{s_k(y_k) y_k^{-1} - 1} \\ T_k y_j &= y_j T_k \text{ for } j \neq k, k+1 \\ y_i y_j &= y_j y_i \\ T_i T_j &= T_j T_i \text{ if } |i-j| > 1 \\ T_k T_{k+1} T_k & \quad T_{k+1} T_k T_{k+1} \text{ for } 1 \leq k \leq n-1 \end{aligned}$$

References

[1] R. Howe, Affine-like Hecke algebras and p-adic representation theory, in "Iwahori-Hecke algebras and their representation theory", *Eds. M. Welleda Baldoni and D. Barbasch*, Martina Franca, Italy 1999, 27 − 69.