Joel's lecture #2

Recall: defined cat-l $\mathfrak{sl}_2$-actions: • sequence of cat-s $\mathcal{D}_r$

• functors $E: \mathcal{D}_r \to \mathcal{D}_{r+2}$, $F: \mathcal{D}_r \to \mathcal{D}_{r-2}$

• nat-l transforms $X: E \to E[2]$, $t: E^2 \to E^2[-2]$

• biadjunctions between E, F

$$EF|_{\mathcal{D}_r} = FE|_{\mathcal{D}_r} \oplus I_{\mathcal{D}_r}^{\oplus r}, \quad r \geq 0$$

$X, t \rightsquigarrow$ endomorphisms $X_1, \dots X_n, t_1, \dots t_{n-1}$ of E^n.

~~Example $\mathcal{D}_r$~~ $\mathfrak{sl}_2$-action $\rightsquigarrow T: \mathcal{D}_r \to \mathcal{D}_{-r}$ defined using complex

$$\dots \to F^{(r+2)} E^{(2)} \to F^{(r+1)} E \to F^{(r)}$$

Example: $\mathcal{D}_r = \mathcal{D}Coh(T^* G(k,n))$, $r = n-2k$

$\rightsquigarrow \mathcal{D}Coh(T^* G(k,n)) \xrightarrow{\sim} \mathcal{D}Coh(T^* G(n-k,n))$

$Z = T^* G(k,n) \times_{B_k} T^* G(n-k,n)$ - equiv-e rel-d to geometry.

Today: simpler cat-l $\mathfrak{sl}_2$-action

$\mathcal{D}_r = \mathcal{D}(\mathcal{D}_{G(k,n)}\text{-mod})$

$I^p(k,n) = \{0 \subset \underset{k}{V} \subset \underset{k+p}{W} \subset \mathbb{C}^n\} \subset G(k,n) \times G(k+p,n)$

$$\begin{tikzcd} & I^p(k,n) \arrow[dl] \arrow[d] \\ G(k,n) & G(k+p,n) \end{tikzcd}$$

$\rightsquigarrow E^{(p)}: \mathcal{D}(\mathcal{D}_{G(k,n)}\text{-mod}) \longrightarrow \mathcal{D}(\mathcal{D}_{G(k+p,n)}\text{-mod})$

w. kernel $\mathcal{S}_{I^p(k,n)} \in \mathcal{D}(\mathcal{D}_{G(k,n)} \times \mathcal{D}_{G(k+p,n)}\text{-mod})$

$\quad \| \quad$

$\iota_* \mathcal{O}_{I^p(k,n)} \qquad \iota: I^p(k,n) \hookrightarrow G(k,n) \times G(k+p,n)$

$F^{(p)}$ - the functor w. same kernel in the other direction

Thm: This gives a categorical $\mathfrak{sl}_2$-action

To define $X: E \to E[2]$ consider line bundle W/V on $I(k,n)$

Get morphism $L_{W/V}: \mathcal{O}_{I(k,n)} \to \mathcal{O}_{I(k,n)}[2] \rightsquigarrow$ morphism $\mathcal{S}_{I^p(k,n)} \to \mathcal{S}_{I^p(k,n)}[2]$

It's pretty easy to see this gives an action of NH_n.

This gives equivalence $T: \mathcal{D}(\mathcal{D}_{G(k,n)}\text{-mod}) \xrightarrow{\sim} \mathcal{D}(\mathcal{D}_{G(n-k,n)}\text{-mod})$

$T = \quad \dots \to \Theta_1 \to \Theta_0, \quad \Theta_s = F^{(r+s)} E^{(s)}[-s]$

1'

Thm: 1) $\Theta_s = IC_{Y_s}[-s]$, where Y_s is as follows:

$G_n \curvearrowright G(k,n)\times G(n-k,n) \rightsquigarrow$ orbit decomp-n $Y_0 \cup \ldots \cup Y_k$, $Y_s = \{(V,W) \mid \dim V\cap W = s\}$

$Z_s = \overline{T^*_{Y_s}(G(k,n)\times G(n-k,n))}$,

$Y_s \rightsquigarrow IC_{Y_s}$ - corresponding to trivial local system on Y_s

Proof of (1): $\exists$ small resolution $P_s \xrightarrow{\pi} \overline{Y}_s = \{(V,W) \mid \dim V\cap W \geq s\}$

$$P_s = \left\{ \begin{tikzcd}[column sep=small] & \mathbb{C}^n \arrow[dl] \arrow[dr] & \\ V \arrow[dr] & & W \arrow[dl] \\ & U - \dim s & \end{tikzcd} \right\}$$

Then $\Theta_s = \pi_* \mathcal{O}_{P_s} = [\text{small resolution}] = IC_{Y_s}[-s]$

$Y_0 = \{(V,W) : V\cap W = 0\}$ - open orbit in $G(k,n)\times G(n-k,n)$

$\overline{Y}_1 = (G(k,n)\times G(n-k,n)) \setminus Y_0$ is a divisor

$j: Y_0 \hookrightarrow G(k,n)\times G(n-k,n)$

Thm 2: Kernel of T is $j_* \mathcal{O}_{Y_0}$

So Θ is obtained from $\begin{tikzcd}[column sep=small] & Y_0 \arrow[dl] \arrow[dr] & \\ G(k,n) & & G(n-k,n) \end{tikzcd}$

Stated w/o proof by Chuang-Rouquier, rel-d: Webster-Williamson

to appear in Cautis-Dodd-Kamnitzer.

Q: How to relate 2-categorical actions • D-modules on $G(k,n)$'s

• Coh sheaves on $T^*G(k,n)$

X-smooth variety $\rightsquigarrow \mathcal{D}_{X,\hbar}$ - $\mathbb{C}[\hbar]$-sheaf of algebras generated by functions, vector fields on X w. rel-ns $[v,f] = \hbar v.f$, $[v_1,v_2]_{\mathcal{D}_{X,\hbar}} = \hbar [v_1,v_2]$.

+ other rel-ns

We have:

$$\mathcal{D}_{X,\hbar} \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1 = \mathcal{D}_X, \quad \mathcal{D}_{X,\hbar} \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_0 = \mathcal{O}_{T^*X}$$

$$\rightsquigarrow \begin{tikzcd}[column sep=small] & \mathcal{D}_{X,\hbar}\text{-mod} \arrow[dl, "\mathbb{C}_0\otimes"'] \arrow[dr, "\mathbb{C}_1\otimes"] & \\ \mathcal{O}_{T^*X}\text{-mod} & & \mathcal{D}_X\text{-mod} \end{tikzcd}$$

Recall: $\mathcal{D}_X$ has filtr-n: by order of diff. op-r: $\mathcal{D}^0_X \subset \mathcal{D}^1_X \subset \ldots$

$\rightsquigarrow \mathcal{D}_{X,\hbar} = \mathrm{Rees}(\mathcal{D}_X) = \bigoplus_k \hbar^k \mathcal{D}^k_X \subset \mathcal{D}_X[\hbar]$

Similarly, if M is a filtered $\mathcal{D}_X$-module, $M^0 \subset M^1 \subset \dots$ w $\mathcal{D}^i M^j \subset M^{i+j}$
Then $Rees(M) = \bigoplus_k \hbar^k M^k$ is $\mathcal{D}_{X,\hbar}$-module
Lem (Laumon, Cautis-Dodd-Kamnitzer) We have kernels & their compositions (e.g. $\mathcal{D}(\mathcal{D}_{X\times Y,\hbar}\text{-mod}) \times \mathcal{D}(\mathcal{D}_{Y\times Z,\hbar}\text{-mod}) \to \mathcal{D}(\mathcal{D}_{X\times Z,\hbar}\text{-mod})$). The functors $\mathbb{C}_0 \otimes_{\mathbb{C}[\hbar]} \cdot$; $\mathbb{C}_1 \otimes_{\mathbb{C}[\hbar]} \cdot$ intertwine composition of kernels

Rem: $f: X \to Y$, $M \in \mathcal{D}(\mathcal{D}_{X,\hbar}\text{-mod}) \rightsquigarrow M \otimes^L_{\mathbb{C}[\hbar]} \mathbb{C}_c$, $(f_* M) \otimes^L_{\mathbb{C}[\hbar]} \mathbb{C}_0$

$$\begin{tikzcd} \mathcal{D}_{X,\hbar}\text{-mod} \arrow[r, "f_*"] \arrow[d] & \mathcal{D}_{Y,\hbar}\text{-mod} \arrow[d] \\ \mathcal{O}_{T^*X}\text{-mod} \arrow[r, dashed] & \mathcal{O}_{T^*Y}\text{-mod} \end{tikzcd}$$

We can define categorical $\mathfrak{sl}_2$-action using $\mathcal{D}_{X,\hbar}$-modules as follows:

$\mathcal{E}^{(p)} = \delta_{I^p(\xi n),\hbar} = \iota_* \mathcal{O}_{I^p(\xi n)}[\hbar]$, $\mathcal{F}^{(p)}$-similar

Thm (Cautis-Dodd-Kamnitzer)

This gives a cat-l $\mathfrak{sl}_2$-action that recovers cat-l $\mathfrak{sl}_2$-actions introduced before.

Now let's identify the kernel of equivalence T.
$\mathcal{D}$-module world suggests $j_*(\mathcal{O}_{Y_0,\hbar})$. But this isn't correct.
E.g. $U = \mathbb{C}^\times$, $X = \mathbb{C}$

$\mathcal{O}_{U,\hbar} = \mathbb{C}[x, x^{-1}, \hbar] \curvearrowleft \mathcal{D}_{U,\hbar}$

$\mathcal{D}_{U,\hbar} = \mathbb{C}\langle x, x^{-1}, \partial, \hbar\rangle / \text{rel-ns}$

$\partial x^n = \hbar n x^{n-1}$

$j_* \mathcal{O}_{U,\hbar} = \mathbb{C}[x, x^{-1}, \hbar] \curvearrowleft \mathcal{D}_{X,\hbar}$ - not finitely generated

Saito: there is a better push-forward

$j_*^{Saito} \mathcal{O}_{U,\hbar}$. In example, get $\mathbb{C}[x,\hbar] \oplus \mathrm{Span}_{\mathbb{C}[\hbar]}(\hbar^{k-1} x^{-k}, k > 0)$
- generated as a $\mathcal{D}_{X,\hbar}$-module by x^{-1}.

U is complement of divisor (i.e $U \hookrightarrow X$ is affine)

Facts: $U \subset X$ - open $(j_*^{Saito} \mathcal{O}_{U,\hbar})/(\hbar=0)$ is finitely generated.
$j_*^{Saito}(\mathcal{O}_{U,\hbar})$ carries a filtration by $\mathcal{D}_{X,\hbar}$-submodules, the associated graded is semisimple object, i.e. $MHM(X) \to \mathcal{D}_{X,\hbar}$-mod (s/simple in $MHM(X)$)

Theorem: The kernel $T \in \mathcal{D}(\mathcal{D}_{G(k,n)\times G(n-k,n),\hbar}$-mod$)$ is $j_*^{Saito}\mathcal{O}_{Y_0,\hbar}$. Moreover, for $s=0\dots k$, $gr_s^W(j_*^{Saito}\mathcal{O}_{Y_0}) = IC_{Y_s,\hbar}$.

Corollary $(j_*^{Saito}\mathcal{O}_{Y_0,\hbar})/(\hbar) = \tilde{j}_* L$
$\tilde{j}: Z^\circ \hookrightarrow Z$
$\{(\overset{''}{X},V,W) \mid \dim \ker X + \dim V \cap W \leq n+1\}$

Generalizations

1) Replace $\mathfrak{sl}_2$ by any symmetric KM algebra
$T^*G(k,n) \rightsquigarrow$ Nakajima quiver variety
$v, w \in \mathbb{Z}^I_{\geq 0}$, I - Dynkin diagram
$\rightsquigarrow M(v,w) = T^*R(v,w) /\!/\!/ GL(v)$
$R(v,w)$ = space of reps of framed quiver Q w. dimension v & framing w
$GL(v) = \prod_i GL(v_i)$
e.g. $M(k,n) = T^*G(k,n)$
Thm (Cautis-Licata-Kamnitzer) + Cautis For fixed w, $\exists$ cat-l $\mathfrak{g}$-action on $(DCoh(M(v,w))_v)$ (modulo KLR rel-ns)
This gives an action of braid grp $B_{\mathfrak{g}} = \langle s_i, i \in I \mid \underbrace{s_i s_j \dots}_{m_{ij}} = \underbrace{s_j s_i \dots}_{m_{ij}} \rangle$
~~which extends t~~ on $D^b(Coh(\bigsqcup_v M(v,w))$
which extends to an action of affine braid group action gen-d by $s_i, Y_i, i \in I$, w. rel-ns on s as before, Y's commute & $s_i Y_j = Y_j s_i$, $i \neq j$ & $s_i = (\prod_{\substack{j \neq i \\ j,i \text{ connected}}} Y_j^{-1}) Y_i s_i^{-1} Y_i$

The s_i's are equivalences coming from each categorical $\mathfrak{sl}_2$-actions & Y_i's are given by line bundles

Generalization of $\mathcal{D}$-module side: $M(v,w)$ has quantization $A(v,w)$ -
- sheaf of alg-s on $M(v,w)$ w. filtrations whose assoc. graded is $\mathcal{O}_{M(v,w)}$
- constructed using quantum Hamilt. red-n: $A(v,w) = \mathcal{D}_{R(v,w)} /\!/ GL(v)$

Thm [Webster, Zheng, Rouquier]
Varagnolo-Vasserot

There is a cat-l $\mathfrak{g}$-action on $\bigoplus_v \mathcal{D}^b(A(v,w)\text{-mod})$

Would like to relate coherent sheaves and modules over quantizations

$A(v,w)\text{-mod} = G\text{-equiv. } \mathcal{D}_{R(v,w)}\text{-mod}^G / I \longleftarrow$ Serre subcategory
(all modules w. sing. supp $\subset$ unstable locus)

Q: $\mathcal{O}_{M(v,w)} \xleftarrow{\hbar=0} G\text{-equiv. } \mathcal{D}_{R(v,w),\hbar}\text{-mod}^G / I_\hbar \xrightarrow{\hbar=1} A(v,w)\text{-mod}$

Gen-n 2: $T^*G(k,n) \rightsquigarrow T^*(G/P)$, G/P is cominuscule flag variety

E.g. $G = SO(2n)$

G/P = quadric in $\mathbb{P}^{2n-1}$

$\to LG(n, 2n) = \{0 \subset L \subset \mathbb{C}^{2n}\}$ lagrangian

$= LG^+ \sqcup LG^-$, $LG^+ = G/P_{n-1}$, $LG^- = G/P_n$

Reason why consider cominuscule flag variety

$G/P \times G/Q$, $Q = w_0 P w_0^{-1}$

G-orbits $\xleftrightarrow{\sim} W_Q \backslash W / W_P$ - linearly ordered for cominuscule P.

So $G/P \times G/Q = Y_0 \cup \ldots \cup Y_k$, $\overline{Y_s} = Y_0 \cup \ldots \cup Y_s$ & $\overline{Y_1}$ is divisor

$\rightsquigarrow Z = T^*G/P \times_B T^*G/Q \rightsquigarrow Z = Z_0 \cup \ldots \cup Z_k$ w $Z_s = \overline{T^*_{Y_s}(G/P \times G/Q)}$

Conj: 1) $G/P \xleftarrow{\ Y_0\ } \to G/Q$ gives equiv. on $\mathcal{D}$-module level

(2) $j_*^{Saito}\mathcal{O}_{Y_0,\hbar}$ is the kernel of an equiv. between $D(\mathcal{D}_{G/P,\hbar}\text{-mod}) \xrightarrow{\sim}$ $D(\mathcal{D}_{G/Q,\hbar}\text{-mod})$

(3) $g^{\vee}{}_s{}^{w}(j_*^{Saito}\mathcal{O}_{Y,\hbar}) = IC_{Y_s,\hbar}$.

(4) $\Theta_s = IC_{Y_s,\hbar} \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_0$ is supported on Z_s.

(5) Cor. of (2) mod $\hbar$.

(6) $\exists$ unique complex using Θ_s and T is the cone

Rem: Y_s no longer seems to have a small resolution