

INTRODUCTION TO GEOMETRIC INVARIANT THEORY

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ABSTRACT. These are the expanded notes for a talk at the MIT/NEU Graduate Student Seminar on Moduli of sheaves on K3 surfaces. We give a brief introduction to GIT, following mostly [N].

1. REDUCTIVE ALGEBRAIC GROUPS.

We are interested in studying orbit spaces for the action of an algebraic group G on a projective or affine algebraic variety X . If we want the orbit space to be an algebraic variety, we have to impose some conditions on the action. For example, if X is affine then it is reasonable to require that the ring of invariants $\mathbb{C}[X]^G$ is finitely generated. Here we introduce some conditions on the group G that ensure that this is always the case.

Definition 1.1. *An algebraic group G is said to be linearly reductive if whenever we have rational representations V, W of G and an epimorphism $\varphi : V \rightarrow W$, the induced morphism on invariants $\varphi^G : V^G \rightarrow W^G$ is also an epimorphism.*

We remark that, over \mathbb{C} (and more generally over any algebraically closed field of zero characteristic) the notion of linear reductivity coincides with the classical notion of reductivity, that is, the radical (= maximal connected normal solvable subgroup) is a torus. In view of this, we will refer to linearly reductive groups simply as *reductive*. In particular finite groups, $\mathrm{GL}(n)$, tori and the classical groups $\mathrm{O}(n)$, $\mathrm{Sp}(2n)$ are all linearly reductive. This stems from the following result.

Lemma 1.2. *Let G be an algebraic group over the field \mathbb{C} . The following are equivalent.*

- (a) G is linearly reductive.
- (b) Every rational representation of G is completely reducible.
- (c) For every epimorphism $\varphi : V \rightarrow W$ of finite dimensional representations of G , the induced map on invariants is an epimorphism.
- (d) For every finite dimensional representation V and every surjective and G -invariant map $\varphi : V \rightarrow \mathbb{C}$ there exists $w \in V^G$ such that $\varphi(w) \neq 0$.
- (e) For every finite dimensional representation V and every nonzero G -invariant vector $v \in V^G$ there exists a nonzero G -invariant map $\varphi : V \rightarrow \mathbb{C}$ with $\varphi(v) \neq 0$.

Proof. (b) \Rightarrow (a) is obvious. For (a) \Rightarrow (b), let V be a representation of G and let $W \subseteq V$ be a subrepresentation. We have an epimorphism $\mathrm{Hom}_{\mathbb{C}}(V, W) \twoheadrightarrow \mathrm{Hom}_{\mathbb{C}}(W, W)$ that by (a) restricts to an epimorphism $\mathrm{Hom}_G(V, W) \twoheadrightarrow \mathrm{Hom}_G(W, W)$. So the inclusion $W \hookrightarrow V$ splits and we are done.

Now, (a) \Leftrightarrow (c) is clear since we are dealing with rational representations of G , and (d) \Leftrightarrow (e) by duality. It is clear that (c) \Rightarrow (d). Let us do (e) \Rightarrow (c). So let V, W be as in the statement of (c) and pick a nonzero $w \in W^G$. By assumption, we can find a G -invariant map $W \rightarrow \mathbb{C}$ with $\varphi(w) \neq 0$. In particular, φ is surjective and so is $\bar{\varphi} : V \rightarrow \mathbb{C}$, the composition of φ with the projection $V \twoheadrightarrow W$. Recalling that (d), (e) are equivalent, we may find $v \in V^G$ with $\bar{\varphi}(v) = \varphi(w)$. So $w - \bar{v} \in \ker \varphi$. If this element is 0, we are done. If not, by induction on $\dim W$ we may find $v_1 \in V^G$ with $\bar{v}_1 = w - \bar{v}$, and the result follows. \square

We remark that over a field of characteristic $p > 0$ the notion of linear reductivity is strictly stronger than that of reductivity. We will comment more about this at the end of this section.

Example 1.3. *The additive group \mathbb{G}_a is not linearly reductive. Indeed, the \mathbb{G}_a -representation on \mathbb{C}^2 given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is not completely reducible.*

The next result, due to Hilbert, justifies the importance of reductive groups in geometric invariant theory.

Theorem 1.4. *Let G be a reductive group acting on an affine algebraic variety X . Then, the algebra of invariants $\mathbb{C}[X]^G$ is finitely generated.*

Proof. First we reduce to the case when $X = V$, a representation of G . Let f_1, \dots, f_ℓ be generators of the algebra of regular functions $\mathbb{C}[X]$. Let $V^* \subseteq \mathbb{C}[X]$ be a finite dimensional G -stable vector space containing these generators. This defines a G -equivariant surjection $\mathbb{C}[V] \rightarrow \mathbb{C}[X]$ that, thanks to the linear reductivity of G restricts to a surjection $\mathbb{C}[V]^G \rightarrow \mathbb{C}[X]^G$. So we only need to show that $\mathbb{C}[V]^G$ is finitely generated. We remark that the action of G on V commutes with the action of \mathbb{C}^\times by dilations. This implies, in particular, that the invariant ring is graded, $\mathbb{C}[V]^G = \bigoplus_{n \geq 0} \mathbb{C}[V]^G_n$. Let $\mathbb{C}[V]^G_+ := \bigoplus_{n > 0} \mathbb{C}[V]^G_n$ and let $J = \mathbb{C}[V]\mathbb{C}[V]^G_+$. By the Hilbert basis theorem, there exist $g_1, \dots, g_k \in \mathbb{C}[V]^G_+$ that generate J as an ideal of $\mathbb{C}[V]$. We may assume the g_i 's are homogeneous. We claim that g_1, \dots, g_k are generators of $\mathbb{C}[V]^G$. Let $h \in \mathbb{C}[V]^G$ be homogeneous. We show by induction on the degree of h that it belongs to $\mathbb{C}[g_1, \dots, g_k]$. If $\deg h = 0$ there is nothing to do. If $\deg h > 0$ then $h \in J$. Note that J is G -stable and so it makes sense to say that $h \in J^G$. The fact that g_1, \dots, g_k generate J as an ideal of $\mathbb{C}[V]$ means that the G -equivariant map

$$\begin{aligned} \mathbb{C}[V]^{\oplus k} &\rightarrow J \\ (h_1, \dots, h_k) &\mapsto \sum_{i=1}^k h_i g_i \end{aligned}$$

is surjective. Since G is linearly reductive, the induced map on invariants is surjective as well. So there exist $h_1, \dots, h_k \in \mathbb{C}[V]^G$ such that $h = \sum_{i=1}^k h_i g_i$. Now the h_i 's are sums of homogeneous elements of degree $< \deg h$. We are done. \square

Let us now mention an important property of reductive groups. It is known that every reductive group contains a (unique up to conjugation) maximal compact *Lie* subgroup K . For example, if $G = \mathrm{GL}(n)$, then $K = \mathrm{U}(n)$, the group of unitary transformations with respect to some invariant scalar product on \mathbb{C}^n . If K' is another compact subgroup, then \mathbb{C}^n admits the structure of a unitary representation of K' , so with respect to some basis of \mathbb{C}^n we have $K' \subseteq \mathrm{U}(n)$. Moreover, if T is a maximal torus inside G then we have $G = KTK$, this is known as the *Cartan decomposition* of G . This is easy to see in the case of $\mathrm{GL}(n)$, where T is the group of diagonal nonsingular matrices with respect to some basis of \mathbb{C}^n .

We finish this section with some comments on what happens over an algebraically closed field \mathbb{K} of characteristic $p > 0$. As we have pointed out before, here the notion of linear reductivity is strictly stronger than that of reductivity. However, reductive groups satisfy a weaker condition, which is a straightforward generalization of Condition (e) in Lemma 1.2.

Definition 1.5. *An algebraic group G is said to be geometrically reductive if, for every representation V of G and every nonzero $v \in V^G$, there exists a G -invariant homogeneous polynomial $f \in \mathbb{C}[V]^G_k$ with $k > 0$ and $f(v) \neq 0$.*

We remark that, in characteristic 0, the notions of linear and geometric reductivity are equivalent (and they are equivalent to the usual notion of reductivity.) In positive characteristic we have: G is linearly reductive $\Rightarrow G$ is geometrically reductive $\Leftrightarrow G$ is reductive. The last part is a theorem of Haboush, see e.g. [MFK, Appendix to Chapter 1]. It follows that classical groups are geometrically reductive. It turns out that geometric reductivity is enough to show many of the results in geometric invariant theory, in particular Theorem 1.4, see e.g. *loc. cit.* We will not go into this.

2. SEMISTABILITY

2.1. Categorical quotients. From now on we will always assume that G is a reductive algebraic group. We have seen that if G acts on an affine variety X , then the algebra of invariants $\mathbb{C}[X]^G$ is finitely generated. So it defines an algebraic variety.

Definition 2.1. *The variety $X//G := \mathrm{Spec} \mathbb{C}[X]^G$ is called the categorical quotient of X by G .*

We have a natural map $\pi : X \rightarrow X//G$, which is clearly constant on orbits. We remark, however, that this map does not separate orbits. Indeed, the map is continuous, so it sends the closure of an orbit to a single point, and orbits are not necessarily closed. However, the next best thing happens.

Lemma 2.2. *Let $\mathbb{O}_1, \mathbb{O}_2 \subseteq X$ be G -orbits. The following are equivalent.*

- (a) $\overline{\mathbb{O}_1} \cap \overline{\mathbb{O}_2} = \emptyset$.
- (b) *There exists $f \in \mathbb{C}[X]^G$ with $f|_{\mathbb{O}_1} \equiv 1$, $f|_{\mathbb{O}_2} \equiv 0$.*

Proof. (b) \Rightarrow (a) is clear. We show (a) \Rightarrow (b). Let $\mathfrak{m}_1, \mathfrak{m}_2$ be the ideals of functions vanishing on $\overline{\mathbb{O}_1}, \overline{\mathbb{O}_2}$, respectively. Since the closures $\overline{\mathbb{O}_1}, \overline{\mathbb{O}_2}$ are G -stable, so are the ideals $\mathfrak{m}_1, \mathfrak{m}_2$. By our assumption, the map $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \rightarrow \mathbb{C}[X]$, $(f_1, f_2) \mapsto f_1 + f_2$ is surjective. Since G is linearly reductive, it follows that we can find $f_1 \in \mathfrak{m}_1^G = \mathfrak{m}_1 \cap \mathbb{C}[X]^G$, $f_2 \in \mathfrak{m}_2^G = \mathfrak{m}_2 \cap \mathbb{C}[X]^G$ with $f_1 + f_2 = 1$. The function f_2 satisfies the requirement of (b). \square

Corollary 2.3. *Let G be a linearly reductive group acting on the affine algebraic variety X . Then, the closure of every orbit \mathbb{O} contains a unique closed orbit. Moreover, this is the orbit of minimal dimension inside $\overline{\mathbb{O}}$.*

Example 2.4. *Let us see that Lemma 2.2 may fail if G is not linearly reductive, even if the algebra $\mathbb{C}[X]^G$ is finitely generated. Take the additive group \mathbb{G}_a acting on \mathbb{C}^2 as in Example 1.3, that is, $t.(x, y) = (x, y + tx)$. Note that every orbit is closed. Indeed, these are the lines $x = c$ for $c \neq 0$ and every point of the form $(0, y)$ is a single orbit. However, the invariant ring is $\mathbb{C}[x] \subseteq \mathbb{C}[x, y]$, which fails to separate closed orbits of the form $(0, y)$.*

Let us mention an important result that will be useful later. Right now, we are in a position to prove one direction in a very easy way. The other direction is considerably harder.

Lemma 2.5 (Matsushima's Criterion). *Let G be an algebraic group and let $H \subseteq G$ be a closed subgroup. Then, G/H is affine if and only if H is reductive.*

Proof. We only prove the 'if' part. Consider the action of H on G by right multiplication, so that the cosets in G/H are precisely orbits for this action. Since H is closed and reductive, Lemma 2.2 implies that $G/H = \text{Spec}(\mathbb{C}[G]^H)$, so it is indeed an affine variety. \square

The following results give us a few more properties of the map π .

Proposition 2.6. *The map π is surjective.*

Proof. Let $y \in X//G$. Let $\mathfrak{m}_y \subseteq \mathbb{C}[X]^G$ be the maximal ideal of the point y , and let $f_1, \dots, f_m \in \mathfrak{m}_y$ be generators. It is easy to see from Definition 1.1 that $\mathfrak{n} := \sum_{i=1}^m \mathbb{C}[X]f_i$ is a proper ideal in $\mathbb{C}[X]$. So any point defined by a maximal ideal lying above \mathfrak{n} maps to y under the map π . We are done. \square

Proposition 2.7. *Let $Z \subseteq X$ be a G -invariant closed subset of X . Then $\pi(Z) \subseteq X//G$ is closed.*

Proof. Let \mathfrak{m} be the ideal of functions vanishing on Z . Since Z is G -invariant, G acts on \mathfrak{m} and so it acts on the quotient ring $\mathbb{C}[X]/\mathfrak{m} = \mathbb{C}[Z]$. Since G is reductive, the surjection $\mathbb{C}[X] \rightarrow \mathbb{C}[X]/\mathfrak{m}$ induces an isomorphism $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[X]^G/\mathfrak{m}^G \xrightarrow{\cong} (\mathbb{C}[X]/\mathfrak{m})^G = \mathbb{C}[Z//G]$. Geometrically, we have that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\pi_Z} & Z//G \\ & \searrow \pi & \swarrow \cong \\ & \pi(Z) & \end{array}$$

The map π_Z is a surjection thanks to Proposition 2.6. The result follows. \square

Remark 2.8. *Note that from the proof of the previous proposition it also follows that $\pi(Z) \cong Z//G$.*

To conclude, we have proved the following properties of π .

Theorem 2.9. *Let X be an affine variety and G a linearly reductive group acting on X . Let $\pi : X \rightarrow X//G$ be the quotient map. The following is true.*

- (a) π is surjective.
- (b) *Every fiber of π contains a unique closed orbit.*

(c) If $Z \subseteq X$ is a closed, G -stable subvariety then $\pi(Z) \subseteq X//G$ is closed and $\pi(Z) \cong Z//G$.

Let us turn those properties of π into a definition.

Definition 2.10. Assume that an algebraic group G acts on a (not necessarily affine) variety X . A good quotient of X by G consists of a pair $(Y, \pi : X \rightarrow Y)$ where Y is a variety and π is an affine morphism satisfying:

- (1) π is G -invariant.
- (2) π is surjective.
- (3) For every open affine $U \subseteq Y$, $\pi^* : \mathbb{C}[U] \rightarrow \mathbb{C}[\pi^{-1}(U)]$ is an isomorphism of $\mathbb{C}[U]$ onto $\mathbb{C}[\pi^{-1}(U)]^G$.
- (4) For every closed invariant subset $V \subseteq X$, we have that $\pi(V)$ is closed in Y .
- (5) If V_1, V_2 are closed, disjoint and G -invariant then $\pi(V_1) \cap \pi(V_2) = \emptyset$.

2.2. Stable points: the affine case. In this subsection we still assume that X is an affine algebraic variety on which the linearly reductive group G acts. As we have seen, the variety $X//G$ parametrizes closed G -orbits. Then, we introduce the following notion.

Definition 2.11. A point $x \in X$ is said to be stable for the G -action if the following two conditions are satisfied.

- (i) The orbit $Gx \subseteq X$ is closed.
- (ii) The stabilizer group $G_x \subseteq G$ is finite.

We denote the set of stable points by X^s .

Proposition 2.12. Let $Z := \{x \in X : \dim G_x > 0\}$. Then, $X^s = X \setminus \pi^{-1}(\pi(Z))$.

Proof. Let $x \in X$. Suppose first that $\pi(x) \in \pi(Z)$. We show that $x \notin X^s$. If $x \in Z$ then Condition (ii) in Definition 2.11 is not satisfied, so we may assume that $x \notin Z$. Since Z is G -stable, it follows that $\pi^{-1}(\pi(x))$ contains at least two orbits. But this is precisely $\{y \in X : x \in \overline{Gy}\}$. Thus, if the orbit of x is closed we may find $y \in \pi^{-1}(\pi(x))$ such that $G.x \subsetneq \overline{G.y}$ is the unique closed orbit. In particular, $\dim Gx < \dim Gy \leq \dim G$, so Condition (ii) is not satisfied. In any case, we get $x \notin X^s$.

On the other hand, assume $x \notin X^s$. Then one of the conditions (i) or (ii) in Definition 2.11 is not satisfied. If Condition (ii) is not satisfied then $x \in Z$ and we are done. So assume that Condition (i) is not satisfied. Let $\mathbb{O} \subseteq G.x$ be the unique closed orbit inside $G.x$. Then $\dim \mathbb{O} < \dim G.x \leq \dim G$. It follows that $y \in Z$ for any $y \in \mathbb{O}$. But $\pi(y) = \pi(x)$, so $x \in \pi^{-1}(\pi(Z))$. We are done. \square

We remark that, using the notation of the previous proposition, Z is a closed set. Indeed, it is easy to see that for every $i = 0, \dots, \dim G$, $X_i := \{x \in X : \dim G_x \leq i\}$ is open, and $Z = X \setminus X_0$. Since Z is also G -invariant, it follows that X^s is open. Moreover, from Proposition 2.12 we have that $\pi(X^s) = X//G \setminus \pi(Z)$, so from Proposition 2.7 it follows that $\pi(X^s) \subseteq X//G$ is also open. Moreover, $X^s = \pi^{-1}(\pi(X^s))$.

Now assume that $x \in X$ is stable. Let $y \in X$ be such that $\overline{Gx} \cap \overline{Gy} \neq \emptyset$. Since Gx is closed, this is equivalent to saying that $Gx \subseteq \overline{Gy}$. So Gx is the unique closed orbit inside \overline{Gy} and therefore its dimension is minimal among the dimensions of the orbits contained in \overline{Gy} . But, since x is stable, $\dim Gx = \dim G$. It follows that $Gx = Gy$. Thus, we get the following result.

Proposition 2.13. The map $\pi : X^s \rightarrow \pi(X^s)$ gives a bijection between $\pi(X^s)$ and the set of G -orbits in X^s .

Definition 2.14. Let X be a (not necessarily affine) variety acted on by the algebraic group G . A geometric quotient of X by G is a good quotient which is also an orbit space, that is, the map π in the definition of a good quotient separates orbits.

So we see that the map $\pi|_{X^s}$ is a geometric quotient. There are a few problems, though. Stable points may not exist, even in the simplest example.

Example 2.15. Consider the action of $G = \mathbb{C}^\times$ on $X = \mathbb{C}^n$ by $t.x = t^{-1}x$. Clearly, $\mathbb{C}[X]^G = \mathbb{C}$, so $X//G = \text{pt}$, and $X^s = \emptyset$.

2.3. Linear actions: the projective case. *From now on, we will assume that X is a projective algebraic variety.* Let us assume that the reductive group G acts on X , and let us fix a closed embedding $X \hookrightarrow \mathbb{P}^n$.

Definition 2.16. *A linearization of the G -action on X is a linear action of G on \mathbb{C}^{n+1} inducing the original action on X . A linear action of G on X is an action together with a chosen linearization.*

We remark that, if we do not fix the embedding $X \hookrightarrow \mathbb{P}^n$, then a linear action of G on X amounts to choosing a G -equivariant very ample line bundle θ on X . A linear action of G on X clearly induces an action of G on $\mathbb{C}[x_0, x_1, \dots, x_n]$. Here, x_0, \dots, x_n are the standard coordinate functions in \mathbb{C}^{n+1} . Now let $\tilde{X} \subseteq \mathbb{C}^{n+1}$ be the affine cone that maps to X under the projection $\mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$. By definition, \tilde{X} is stable under the G -action. Since \tilde{X} is a cone, there is a natural \mathbb{C}^\times -action on \tilde{X} and it is clear that the G - and \mathbb{C}^\times -actions on \tilde{X} commute. In more algebraic terms: the algebra $\mathbb{C}[\tilde{X}]$ is graded, with $\mathbb{C}[\tilde{X}]_0 = \mathbb{C}$, and the action of G on $\mathbb{C}[\tilde{X}]$ preserves the degree of an element. In particular, the algebra $\mathbb{C}[\tilde{X}]^G$ is a finitely generated graded algebra. So it makes sense to consider $Y := \text{Proj}(\mathbb{C}[\tilde{X}]^G)$. This is a projective variety that we call the *GIT quotient* of X by G .

Of course, the variety Y depends on the choice of a linearization θ of the G -action. Now we would like to explain in which sense Y is a reasonable quotient of (an appropriate subset of) X by G . To do so, first we recall a few basic facts on the Proj construction.

Lemma 2.17. *The following is true.*

- (i) *Y is covered by affine open sets Y_f for $f \in \mathbb{C}[\tilde{X}]^G$ an homogeneous element of degree ≥ 1 . Moreover, $\mathbb{C}[Y_f] = \mathbb{C}[\tilde{X}]^G[f^{-1}]_0$, the algebra of degree 0 elements on $\mathbb{C}[\tilde{X}]^G[f^{-1}]$.*
- (ii) *Let $X^{\theta-ss} := \bigcup \{X_f : f \in \mathbb{C}[\tilde{X}]^G \text{ is homogeneous of degree } \geq 1\}$. Then, the morphisms $X_f \rightarrow Y_f$ induced by the inclusions $\mathbb{C}[\tilde{X}]^G[f^{-1}]_0 \rightarrow \mathbb{C}[\tilde{X}]^G[f^{-1}]_0$ glue together to give a (by definition, affine) morphism $\phi : X^{\theta-ss} \rightarrow Y$.*

Based on the previous lemma, we make the following definition.

Definition 2.18. *Let G act on the projective variety X and fix a linearization θ of this action. A point $x \in X$ is said to be θ -semistable if there exist $n > 0$ and $f \in \mathbb{C}[\tilde{X}]_n^G$ with $f(x) \neq 0$. Thus, the set of θ -semistable points is precisely the set $X^{\theta-ss}$ from Lemma 2.17.*

So we get a map $\phi : X^{\theta-ss} \rightarrow Y$. Let us examine the properties of this map that follow from the work we have already done with affine varieties.

Proposition 2.19. *The map $\phi : X^{\theta-ss} \rightarrow Y$ is a good quotient of $X^{\theta-ss}$ by G .*

Proof. Recall that the map ϕ is constructed by gluing maps between affine varieties. Moreover, it is clear that for $f \in \mathbb{C}[\tilde{X}]_n^G$, $n > 0$, we have that $\mathbb{C}[Y_f] = \mathbb{C}[\tilde{X}]^G[f^{-1}]_0 = (\mathbb{C}[\tilde{X}][f^{-1}]_0)^G = \mathbb{C}[X_f]^G$. This observation, together with Theorem 2.9 completes the proof. \square

For the affine case, we were able to find an open subset X^s of X such that the restriction of $\pi : X \rightarrow X//G$ behaves like a geometric quotient. We would like to do the same for the projective case. This motivates the following definition.

Definition 2.20. *Fix a linearization θ for the action of G on X . A point $x \in X$ is said to be θ -stable if $\dim G \cdot x = \dim G$, and there exist $n > 0$, $f \in \mathbb{C}[\tilde{X}]_n^G$ with $x \in X_f$ and the action of G on X_f is closed (meaning that all orbits are closed in X_f .) We denote the set of stable points by $X^{\theta-s}$. Clearly, $X^{\theta-s} \subseteq X^{\theta-ss}$.*

We remark that $X^{\theta-s}$ is open in X . Indeed, the set $\{x \in X : \dim G \cdot x = \dim G\}$ is open in X since the dimension of an orbit is a lower semi-continuous function on x .

Proposition 2.21. *There exists an open subset $Y^s \subseteq Y$ such that $\phi^{-1}(Y^s) = X^{\theta-s}$. Moreover, $\phi|_{X^{\theta-s}}$ is a geometric quotient.*

Proof. Let $Y^s := \phi(X^{\theta-s})$. We must show, first, that Y^s is open in Y . In order to do so, let $Y^0 := \bigcup\{Y_f : f \in \mathbb{C}[\tilde{X}]_n^G \text{ for some } n \geq 1 \text{ and the action of } G \text{ on } X_f \text{ is closed}\}$, this is clearly an open subset of Y . We remark that $\phi|_{\phi^{-1}(Y_0)}$ is a geometric quotient - it is a good quotient since this is a local notion, and by the definition of Y^0 it is an orbit space. This already implies that $X^{\theta-s} = \phi^{-1}(Y^s)$ and that $\phi(\phi^{-1}(Y_0) \setminus X^{\theta-s}) = Y_0 \setminus Y_s$. Since $\phi|_{\phi^{-1}(Y_0)}$ is a geometric (in particular, good) quotient, this is a closed set in Y_0 . So $Y^{\theta-s}$ is open in Y_0 and so it is also open in Y . Finally, it is an easy exercise to show that $\phi : X^{\theta-s} \rightarrow Y^s$ is a geometric quotient. \square

So we see that $\phi : X^{\theta-s} \rightarrow Y^s$ is a geometric quotient. By construction, the variety Y is compact and should be thought of as a compactification of the orbit space Y^s . The following is a slightly easier description of stable points.

Proposition 2.22. *For $x \in X^{\theta-ss}$, the following are equivalent.*

- (1) $x \in X^{\theta-s}$.
- (2) *The orbit $G.x$ is closed in $X^{\theta-ss}$ and $\dim G.x = \dim G$.*
- (3) *The orbit $G.x$ is closed in $X^{\theta-ss}$ and the stabilizer G_x is finite.*

Proof. (1) \Rightarrow (2). Assume $x \in X^{\theta-s}$. The only thing we need to show is that $G.x$ is closed in $X^{\theta-ss}$. Well, we have that $\phi^{-1}(\phi(x)) \subseteq X^{\theta-s}$, this follows from the previous proposition. Now, $\phi^{-1}(\phi(x))$ is clearly a closed subset of $X^{\theta-ss}$ and so $\overline{G.x} \cap X^{\theta-ss} \subseteq \phi^{-1}(\phi(x)) \subseteq X^{\theta-s}$. By Proposition 2.21, the action of G on $X^{\theta-s}$ is closed. So $G.x$ is closed in $X^{\theta-s}$. Therefore, $G.x = \overline{G.x} \cap X^{\theta-s} = \overline{G.x} \cap X^{\theta-ss}$. So $G.x$ is closed in $X^{\theta-ss}$.

(2) \Rightarrow (1). First of all, since $x \in X^{\theta-ss}$ there exist $n > 0$ and $f \in \mathbb{C}[\tilde{X}]_n^G$ with $x \in X_f$. Since $G.x$ is closed in $X^{\theta-ss}$, it is also closed in X_f . Now, consider $Z := \{y \in X_f : \dim G.y < \dim G\}$, this is a closed subset of Y that is disjoint from $G.x$. So thanks to -a slight generalization of- Lemma 2.2 we may find $g \in \mathbb{C}[X_f]^G$ with $g(Z) = 0, g(x) = 1$. Recall that, by definition, $\mathbb{C}[X_f] = \mathbb{C}[\tilde{X}][f^{-1}]_0$, so $g = h/f$ for some invariant homogeneous $h \in \mathbb{C}[\tilde{X}]$ (note that here we are using that G is reductive.) Clearly, $x \in X_{fh} \subseteq X_f \setminus Z$. Thus, for every $y \in X_{fh}, \dim G.y = \dim G$. This implies, thanks to Corollary 2.3, that the action of G on X_{fh} is closed. Finally, (2) \Leftrightarrow (3) follows from the arguments in this paragraph and Matsushima's criterion, cf. Lemma 2.5. \square

2.3.1. Example: Quadric hypersurfaces in \mathbb{P}^n . A quadric hypersurface in \mathbb{P}^n is the set of zeroes of an homogeneous polynomial of degree 2 in $\mathbb{C}[x_0, \dots, x_n]$. These can be identified with the projective space $\mathbb{P}(S)$, where S is the set of symmetric $(n+1) \times (n+1)$ -matrices. So we get an action of $\mathrm{SL}(n+1)$ on S by $g.A = (g^t)^{-1}Ag^{-1}$ that descends to an action on $\mathbb{P}(S)$. Let us find stable and semistable points here. First, there is a clear homogeneous invariant, the determinant \det , and we have $\mathbb{P}(S)_{\det} \subseteq \mathbb{P}(S)^{\theta-ss}$. It is clear that $\mathbb{P}(S)_{\det}$ is a *single* $\mathrm{SL}(n+1)$ -orbit. Note, however, that $\dim(\mathbb{P}(S)_{\det}) < \dim \mathrm{SL}(n+1)$, so points in $\mathbb{P}(S)_{\det}$ are *not* stable.

We claim now that if $f \in \mathbb{C}[S]$ is homogeneous and invariant then there exists $\lambda \in \mathbb{C}, k \geq 0$ such that $f = \lambda \det^k$. This follows easily from the fact that $\mathbb{P}(S)_{\det}$ is a single $\mathrm{SL}(n+1)$ -orbit and $\mathbb{P}(S)_{\det}$ is dense in $\mathbb{P}(S)$. So we get that there are no stable quadric hypersurface in $\mathbb{P}(n)$ and a quadric hypersurface is semistable if and only if it is nonsingular. The GIT quotient is $Y = \mathrm{Proj}(\mathbb{C}[\det]) = \mathrm{pt}$.

2.3.2. Example: Binary cubics. A *binary form of degree n* is a homogeneous polynomial of degree n in 2 variables:

$$f = a_0x_0^n + a_1x_0^{n-1}x_1 + a_2x_0^{n-2}x_1^2 + \dots + a_nx_1^n$$

Note that the set of binary forms of degree n forms a vector space V_{n+1} of dimension $n+1$ that we are going to identify with \mathbb{C}^{n+1} . The corresponding projective space \mathbb{P}^n is in correspondence with the set of n points (with multiplicities) on \mathbb{P}^1 : a binary form f corresponds to the roots of the equation $f(x_0, x_1) = 0$. We have a natural action of $\mathrm{SL}(2)$ on \mathbb{P}^1 , that therefore determines an action on \mathbb{P}^n . We linearize this action as follows: $\mathrm{SL}(2)$ acts on V_{n+1} by:

$$g.f(x_0, x_1) = f(g^{-1}(x_0, x_1)) \quad f \in V_{n+1}, g \in \mathrm{SL}(2)$$

We recall that for $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{P}^1$, with the x'_i 's and y'_i 's all distinct, there exists a unique element $g \in \mathrm{PGL}(2)$ with $g(x_i) = y_i$. This fact will be important in our argument. We focus in the case $n = 3$, the so-called *binary cubics*. We have one invariant, the *discriminant*

$$\Delta := 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3$$

The set \mathbb{P}_Δ^3 is the set of all binary cubics without repeated roots. So \mathbb{P}_Δ^3 is a single $\mathrm{SL}(2)$ -orbit. Note that \mathbb{P}_Δ^3 is dense in \mathbb{P}^3 and so by a similar argument to that of Example 2.3.1 we get that every homogeneous invariant has the form $\lambda\Delta^k$ for $\lambda \in \mathbb{C}, k \geq 0$. Thus, $\mathbb{P}_\Delta^3 = (\mathbb{P}^3)^{\theta-ss}$. Moreover, $\dim \mathbb{P}_\Delta^3 = 3 = \dim \mathrm{SL}(2)$, so by Proposition 2.22 we get $\mathbb{P}_\Delta^3 = (\mathbb{P}^3)^{\theta-ss} = (\mathbb{P}^3)^{\theta-s}$. The GIT quotient Y is again a point.

We will examine binary forms of higher degree in the next section, after we have developed more sophisticated tools for finding (semi)stable points.

3. THE HILBERT-MUMFORD CRITERION

So far, the only way we have to find (semi)stable points is to explicitly find invariants and look at their non-vanishing locus, see e.g. Examples 2.3.1, 2.3.2. The goal of this section is to explain a more numerical criterion for computing stability. We start with the following re-statement of the notion of stability. So fix a projective variety $X \hookrightarrow \mathbb{P}^n$, with a linear action of a reductive group G .

Lemma 3.1. *Let $x \in X$ and let $\tilde{x} \in \mathbb{C}^{n+1}$ be a point lying over x . Then:*

- (i) *x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$.*
- (ii) *x is stable if and only if $G \cdot \tilde{x}$ is closed and the stabilizer $G_{\tilde{x}}$ is finite.*

Proof. The proof of (i) is an exercise, while (ii) is a reformulation of Proposition 2.22. □

3.1. The Hilbert-Mumford Criterion. In this subsection we state and prove the Hilbert-Mumford Criterion. This is a technique that allows us to find the locus of (semi)stable points in X by looking at actions of 1-parametric subgroups of G , that is, morphisms $\lambda : \mathbb{C}^\times \rightarrow G$. It is, basically, a consequence of the following theorem.

Theorem 3.2 (Hilbert-Mumford Criterion). *Let G be a reductive algebraic group acting rationally on a vector space V . Let $x \in V$ and let $y \in \overline{Gx}$ be such that Gy is the unique closed orbit in \overline{Gx} . Then, there exists a 1-parametric subgroup $\lambda : \mathbb{C}^\times \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t).x \in Gy$.*

Let us say a few words on the proof of Theorem 3.2. We follow a proof by R. Richardson in [B]. We remark that this proof, while considerably more elementary than the one in [MFK], does not generalize to arbitrary characteristic. We will use the fact that, for a \mathbb{C} -vector space V and a locally closed subset $W \subseteq V$ (for example, an orbit under the action of a reductive group) we have $\overline{W} = \overline{W}^\vee$ where \bullet^\vee denotes the closure in the usual topology. So assume the hypothesis in the statement of Theorem 3.2. First, we reduce to the case where G is a torus. This is done by the following.

Claim: *There exists a maximal torus $T \subseteq G$ such that $G.y \cap \overline{T.x} \neq \emptyset$*

We proceed by contradiction. Assume the claim is not true, that is, $G.y \cap \overline{T.x} = \emptyset$ for every maximal torus T of G . Choose any maximal torus T . For $z = gx \in G.x$, we get $\overline{T.z} = \overline{g(g^{-1}Tg).x}$, so that $G.y \cap \overline{T.z} = g(G.y \cap \overline{(g^{-1}Tg).x}) = \emptyset$ because $g^{-1}Tg$ is a maximal torus. Thus, there exists $f_z \in \mathbb{C}[V]^T$ such that $f_z(G.y) = 0$ and the restriction of f_z to $\overline{T.z}$ is the constant 1. Let $U_z \subseteq V$ be the open set defined by f_z .

Now we pick a Cartan decomposition $G = KTK$ of G . Since $K.x$ is compact and $K.x \subseteq \bigcup_{z \in G.x} U_z$, we have that there exist z_1, \dots, z_k such that $K.x \subseteq \bigcup_{i=1}^k U_{z_i}$. We define the function $f : V \rightarrow \mathbb{R}$ by $f(z) = \sum_{i=1}^k |f_{z_i}(z)|$. Note that this function is continuous in the usual topology, and that $0 \notin f(K.x)$. So f attains a *strictly positive* minimum value on $K.x$. Since f is clearly T -invariant, the same is true for $TK.x$

and $\overline{TK.x}^\vee$. Since $f(G.y) = 0$, we see that $G.y \cap \overline{TK.x}^\vee = \emptyset$, hence $G.y \cap \overline{KTK.x}^\vee = \emptyset$. We claim that this is already a contradiction. Indeed, using that $G = KTK$ we have:

$$G.x = KTK.x \subseteq K(\overline{TK.x}^\vee) \subseteq \overline{KG.x}^\vee = \overline{G.x}^\vee$$

Since K is compact, $\overline{KTK.x}^\vee$ is closed in the usual topology. So from the above chain of inclusions we get $\overline{KTK.x}^\vee = \overline{G.x}^\vee = \overline{G.x}$. So $G.y \cap \overline{G.x} = \emptyset$, a contradiction. Thus, the claim is true.

Now we need to show Theorem 3.2 in the case where $G = T = (\mathbb{C}^\times)^n$, a torus. So let V be a representation of T and let $x \in V$. First, note that we have a weight decomposition for V : there exist $\alpha_1, \dots, \alpha_k \in \mathbb{Z}^n$, not necessarily all distinct, such that:

$$V = \bigoplus_{i=1}^k V_{\alpha_i}$$

Let $e_i \in V_{\alpha_i}$ be nonzero. So we have $x = \sum_{i=1}^k x_i e_i$, and we may assume without loss of generality that $x_i \neq 0$ for all i . We remark that $T.x$ is closed in the set $\{\sum_{i=1}^k y_i e_i \in V : y_i \neq 0 \text{ for all } i\}$, this follows because the image of T under the action map is closed in $\text{GL}(V)$.

Now let $X := \{m = (m_1, \dots, m_k) \in \mathbb{Z}^k : \sum_{i=1}^k m_i \alpha_i = 0\}$, and $X^+ := X \cap \mathbb{Z}_{\geq 0}^k$. For any $m \in X^+$ we have a map $m : V \rightarrow \mathbb{C}$, $\sum v_i e_i \mapsto v_1^{m_1} \cdots v_k^{m_k}$. By definition, $m|_{T.x}$ is a nonzero constant, so $m|_{\overline{T.x}}$ is also that constant. It follows that if there exists $i \in \{1, \dots, k\}$ with $m_i \neq 0$, then for every $y = \sum y_j e_j \in \overline{T.x}$ we must have $y_i \neq 0$. By the observation at the end of the previous paragraph, $\{i \in \{1, \dots, k\} : m_i = 0 \text{ for all } m \in X^+\}$ is nonempty. Changing the order if necessary, we may assume that this set is $\{1, \dots, m\}$ for some $1 \leq m \leq k$.

An exercise now is to check that there exists a linear map $f : \mathbb{Z}^n \rightarrow \mathbb{Z}$ such that $f(\alpha_i) > 0$ if $i = 1, \dots, m$ and $f(\alpha_i) = 0$ if $i = m+1, \dots, k$. A hint: if we denote by W the subgroup generated by $\{\alpha_{m+1}, \dots, \alpha_k\}$, then 0 is not in the convex hull of $\{\alpha_1 + W, \dots, \alpha_m + W\}$ in \mathbb{Z}^n/W . This map determines a 1-parametric subgroup $\lambda : \mathbb{C}^\times \rightarrow (\mathbb{C}^\times)^n$, $t \mapsto (t^{f(\epsilon_i)})_{i=1}^n$, where $\{\epsilon_i\}$ is the tautological basis of \mathbb{Z}^n . Note that, by definition, we get:

$$\lim_{t \rightarrow 0} \lambda(t).x = \lim_{t \rightarrow 0} \sum_{i=1}^k t^{f(\alpha_i)} x_i e_i = \sum_{i=m+1}^k x_i e_i =: y$$

To finish the proof, we show that $T.y$ is closed. If it is not then, repeating the procedure above, we may find a 1-parametric subgroup λ' with $\lim_{t \rightarrow 0} \lambda'(t).y = \sum_{i=m+1}^k x'_i e_i =: z$, with $x'_i = x_i$ or 0 . Since $z \in \overline{G.y} \subseteq \overline{G.x}$, then by our work above we have that $x'_i \neq 0$ for all $i \in \{m+1, \dots, k\}$. So $z = y$. Thus, $T.y$ is closed.

3.2. A numerical criterion. Let us rephrase the Hilbert-Mumford criterion in the way we are going to apply it. To do so, we introduce some notation regarding 1-parametric subgroups. So let G be a reductive algebraic group acting rationally on V and let $\lambda : \mathbb{C}^\times \rightarrow G$ be a 1-parametric subgroup. So we may consider V as a representation of \mathbb{C}^\times . Every such representation V is completely reducible, so there exists a basis e_0, \dots, e_n of V and $r_0, \dots, r_n \in \mathbb{Z}$ such that $\lambda(t).e_i = t^{r_i} e_i$. For a nonzero $v = \sum_{i=0}^n v_i e_i$ define:

$$\mu(v, \lambda) := \max\{-r_i : v_i \neq 0\}$$

Note that by definition we have, for $\mu \in \mathbb{Z}$

$$\lim_{t \rightarrow 0} t^\mu (\lambda(t).v) = \begin{cases} \text{does not exist} & \text{if } \mu < \mu(v, \lambda) \\ \sum_{i=0}^n \delta_{0, \mu(v, \lambda) + r_i} v_i e_i \neq 0 & \text{if } \mu = \mu(v, \lambda) \\ 0 & \text{if } \mu > \mu(v, \lambda) \end{cases}$$

So $\mu(v, \lambda)$ does not depend on the chosen basis. It follows that:

$$(1) \quad \begin{aligned} \mu(v, \lambda) > 0 &\Leftrightarrow \lim_{t \rightarrow 0} \lambda(t).v \text{ does not exist} \\ \mu(v, \lambda) = 0 &\Leftrightarrow \lim_{t \rightarrow 0} \lambda(t).v \text{ exists and it is nonzero} \end{aligned}$$

So an obvious consequence of (1) and Lemma 3.1 is the following.

Proposition 3.3. *Assume a reductive group G acts on the projective variety $X \hookrightarrow \mathbb{P}^n$, and fix a linearization for the action with respect to this inclusion. For $x \in X$, let \tilde{x} be a lift of x to \mathbb{C}^{n+1} . Then, x is semistable if and only if $\mu(\tilde{x}, \lambda) \geq 0$ for every 1-parametric subgroup λ of G .*

Let us proceed to stability.

Proposition 3.4. *With the notation of Proposition 3.3, we have that x is stable if and only if $\mu(\tilde{x}, \lambda) > 0$ for every nontrivial 1-parametric subgroup λ of G .*

Proof. We need to show that x is stable if and only if, for every nontrivial 1-parametric subgroup λ , $\lim_{t \rightarrow 0} \lambda(t).\tilde{x}$ does not exist. First, assume that x is stable. If there exists a 1-parametric subgroup with $y := \lim_{t \rightarrow 0} \lambda(t).\tilde{x}$, then $y \in G.\tilde{x} = \overline{G.\tilde{x}}$ is stabilized by $\lambda(\mathbb{C}^\times)$, contradicting the fact that the stabilizer of \tilde{x} is finite.

Conversely, assume that for every 1-parametric subgroup $\lambda : \mathbb{C}^\times \rightarrow G$, $\lim_{t \rightarrow 0} \lambda(t).\tilde{x}$ does not exist. It follows immediately from the Hilbert-Mumford criterion that $G.\tilde{x}$ is closed, hence affine. By Matsushima's Criterion, $G_{\tilde{x}}$ is reductive. Thus, if it is not finite then, thanks to the Cartan decomposition, it must contain a 1-dimensional torus. But then we can find a 1-parametric subgroup stabilizing \tilde{x} , contradicting our hypothesis. Thus, x is stable. \square

3.3. Example. Binary forms. Let us apply the Hilbert-Mumford criterion to find the (semi)stable binary forms of degree n , see Example 2.3.2. First of all, we remark that every 1-parametric subgroup of $\mathrm{SL}(2)$ is conjugate to one of the form:

$$\lambda_r : t \mapsto \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}, \quad r > 0$$

which can be easily seen from the fact that every representation of \mathbb{C}^\times is completely reducible. So a binary form is not (semi)stable if and only if it is $\mathrm{SL}(2)$ -conjugate to one for which $\mu(f, \lambda_r) \leq 0$ (< 0) for some r . Now, it is obvious that $\lambda_r(t).x_0^i x_1^{n-i} = t^{r(2i-n)} x_0^i x_1^{n-i}$, so the induced action of \mathbb{C}^\times is diagonal with respect to the obvious basis of the space V_{n+1} of binary n -forms. Thus,

$$\mu(f = \sum_{i=0}^n a_i x_0^i x_1^{n-i}, \lambda_r) = r(n - 2i_0),$$

where $i_0 := \min\{i : a_i \neq 0\}$. Thus, we have $\mu(f, \lambda_r) \leq 0$ (< 0) if and only if $a_i = 0$ for $i < n/2$ ($i \leq n/2$), that is, if and only if the point $[0 : 1]$ is a 0 of multiplicity $\geq n/2$ ($> n/2$). So we get.

Proposition 3.5. *A binary n -form is (semi)stable if and only if no point of \mathbb{P}^1 occurs as a point of multiplicity $\geq n/2$ ($> n/2$) for the given form. In particular, if n is odd, a form is semistable if and only if it is stable.*

Note that for the case $n = 3$ this agrees with Example 2.3.2, but has the advantage that we do not need to compute the invariants. For the case $n = 4$, we have that a binary quartic is stable if all of its points are simple, and there are two types of semistable quartics: those with two double points and those with one double point and two simple points. For the stable quartics, we can fix one of the points and can send the other three points to any other three points. So, if $\phi : \mathbb{P}(V_4) \rightarrow Y$ denotes the quotient, we have that $\phi(\mathbb{P}(V_4)^s) = \mathbb{C}$. Similarly, we can see that each type of semistable quartics forms a single $\mathrm{SL}(2)$ -orbit. These two orbits get identified to a single point in the quotient, and $Y = \mathbb{P}^1$.

3.4. Example: N ordered points on a line. Let us describe first a general situation. Let X be a projective variety and fix an inclusion $X \hookrightarrow \mathbb{P}^n$. Assume the reductive group G acts on X and fix a linearization of this action with respect to the fixed inclusion $X \hookrightarrow \mathbb{P}^n$. Consider the diagonal action of G on X^N . We claim that this action has a natural linearization. Indeed, first of all note that X^N is a projective variety via the Segre embedding $(\mathbb{P}^n)^N \hookrightarrow \mathbb{P}^k$, $k = (n+1)^N - 1$. By the very definition of the Segre embedding, the induced action of G on the N -fold tensor product $(\mathbb{C}^{n+1})^{\otimes N}$ is a linearization of the diagonal action of G on X^N .

Now let λ be a 1-parameter subgroup of G . It is an easy exercise to see that, for $x_1, \dots, x_N \in X$:

$$\mu((x_1, \dots, x_N), \lambda) = \sum_{i=1}^N \mu(x_i, \lambda)$$

Let us apply this in a concrete situation. Let $X = \mathbb{P}^1$, $G = \mathrm{SL}(2)$. Using the notation from the previous example, we have that for $x \in \mathbb{P}^1$:

$$\mu(x, \lambda_r) = \begin{cases} r & x \neq [0 : 1] \\ -r & x = [0 : 1] \end{cases}$$

Thus, for $x = (x_1, \dots, x_N) \in (\mathbb{P}^1)^N$, let $q = \#\{i : x_i = [0 : 1]\}$. We have $\mu((x_1, \dots, x_N), \lambda_r) = (N - 2q)r$. Then, we get that a point (x_1, \dots, x_N) is (semi)stable if and only if no coordinate is repeated $\geq (>) N/2$ times. Note, in particular, that if N is odd then every semistable point is stable.

3.5. Example: Plane Cubics. A *plane cubic* is a curve in \mathbb{P}^2 defined by an homogeneous polynomial of degree 3. We assume that this polynomial has the form

$$f = a_{30}x_3^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_3^3 + a_{20}x_0x_1^2 + a_{11}x_0x_1x_2 + a_{02}x_0x_2^2 + a_{10}x_0^2x_1 + a_{01}x_0^2x_2 + a_{00}x_3^3$$

We have an action of $\mathrm{SL}(3)$ on the set of all plane cubics. This action is clearly linear. Now, every 1-parametric subgroup of $\mathrm{SL}(3)$ is conjugate to one of the form:

$$\lambda : t \mapsto \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2}), r_0 \geq r_1 \geq r_2, r_0 + r_1 + r_2 = 0$$

Under such λ , we have that:

$$\mu(f, \lambda) = \max\{(3 - i - j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}$$

So, if $\mu(f, \lambda) < 0$, then $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = 0$. Conversely, if these numbers are 0 then for $r_0 = 3, r_1 = -1, r_2 = -2$ we get $\mu(f, \lambda) < 0$. So we get that a curve is not semistable if and only if $[1 : 0 : 0]$ is a triple point or a double point with a unique tangent. Then, a curve is semistable if and only if it has no triple points and no double points with a unique tangent.

Let us now find the stable locus. First of all, if $[1 : 0 : 0]$ is a singular point then $a_{00} = a_{10} = a_{01} = 0$. Taking now $r_0 = 2, r_1 = -1 = r_2$ we get $\mu(f, \lambda) \leq 0$. So f is not stable. Thus, if f is stable then it has no singular points. Conversely, we check that if there exists λ of the form above such that $\mu(f, \lambda) \leq 0$ then f has a singular point. We have that, if this is the case, then $a_{00} = a_{10} = 0$. If $a_{01} = 0$ then $[1 : 0 : 0]$ is a singular point and we are done. If $a_{10} \neq 0$ then $2r_0 + r_2 \leq 0$, so that $r_2 = -2r_0, r_1 = r_0$ (indeed, we cannot have $2r_0 + r_2 < 0$ because of the requirements $r_0 > r_1 > r_2, r_0 + r_1 + r_2 = 0$.) Thus, $\mu(\lambda, f) = \max\{(3 - 3j)r_0 : a_{ij} \neq 0\}$. Thus, $\mu(f, \lambda) \leq 0$ if and only if $a_{i0} = 0$ for all i . This implies that $f = x_2f'$ for some form f' of degree 2. So f is singular at points for which $x_2 = f' = 0$. To conclude, f is stable if and only if it is nonsingular.

4. LUNA'S SLICE THEOREM

We finish these notes with a brief discussion of the étale slice theorem of Luna and some of its applications. This is a fundamental result that is helpful in studying the étale topology of the affine quotients $X//G$ from Subsection 2.1. We remark that throughout this section all our varieties are defined over \mathbb{C} . Over fields of positive characteristic, the definitions become slightly more complicated.

4.1. Étale Slices. We recall first the notion of an étale morphism. For a variety X and a point $x \in X$, we denote by \mathcal{O}_x then stalk of the structure sheaf at x .

Definition 4.1. *Let X, Y be varieties and let $f : X \rightarrow Y$ be a morphism. For $x \in X$, the morphism f is called étale at x if the map $\mathcal{O}_{f(x)}^\wedge \rightarrow \mathcal{O}_x^\wedge$ is an isomorphism, where \bullet^\wedge denotes the completion at the maximal ideal. We say that the map f is étale if it is étale at every point $x \in X$.*

So a morphism is étale at x if it induces an isomorphism between the formal neighborhoods of x and $f(x)$. For example, an open immersion is étale. We remark that étale morphisms are stable under composition and base change. We remark that, if X is smooth at x and Y is smooth at $f(x)$, then f is étale at x if and only if $T_x f : T_x X \rightarrow T_{f(x)} Y$ is an isomorphism. Now let us bring a G -action into the picture.

Definition 4.2. *Let G be a reductive group; X, Y affine G -varieties and a G -equivariant map $f : X \rightarrow Y$. We say that f is strongly étale if*

- (i) *The induced map $\bar{f} : X//G \rightarrow Y//G$ is étale.*
- (ii) *The G -morphism $f \times \pi_X : X \rightarrow Y \times_{Y//G} X//G$ is an isomorphism.*

In other words, f is strongly étale if the following diagram is Cartesian with the bottom horizontal arrow being étale.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X//G & \xrightarrow{\bar{f}} & Y//G \end{array}$$

Now assume $f : X \rightarrow Y$ is a strongly étale G -morphism. The following are immediate consequences of the definition.

- (1) f is étale and its image is an open subset $U \subseteq Y$ satisfying $U = \pi_Y^{-1}(\pi_Y(U))$.
- (2) For every $u \in X//G$, f induces an isomorphism $\pi_X^{-1}(u) \cong \pi_Y^{-1}(f(u))$.
- (3) For every $x \in X$, $f|_{G_x}$ is injective. Moreover, G_x is closed if and only if $G \cdot f(x)$ is closed.

We are now ready to give the definition of an étale slice. So let X be an affine G -variety and $x \in X$ be such that G_x is closed, in particular G_x is reductive. Let $S \subseteq X$ be a locally closed, G_x -invariant subvariety of X , with $x \in X$. We have a G_x -action on $G \times S$ that is given by $h \cdot (g, s) = (gh^{-1}, hs)$. The multiplication morphism $\mu : G \times S \rightarrow X$, $(g, s) \mapsto gv$ is G_x -invariant and so it induces a morphism of G -varieties:

$$\psi_{S,x} : G \times_{G_x} S := (G \times S)//G_x \rightarrow X$$

where G acts on the domain by left-multiplication on the first component.

Definition 4.3. *S is called an étale slice at x if the map $\psi_{S,x}$ is strongly étale.*

Let us remark that $(G \times_{G_x} S)//G \cong S//G_x$. So, paraphrasing, S is an étale slice if the diagram

$$\begin{array}{ccc} G \times_{G_x} S & \xrightarrow{\psi_{S,x}} & X \\ \downarrow \pi & & \downarrow \pi \\ S//G_x & \xrightarrow{\overline{\psi_{S,x}}} & X//G \end{array}$$

is cartesian and the morphism $\overline{\psi_{S,x}}$ (and therefore also $\psi_{S,x}$) is étale. Thus, we have an isomorphism $\mathcal{O}_{x,S//G_x}^\wedge \cong \mathcal{O}_{x,X//G}^\wedge$.

4.2. The étale slice theorem.

Theorem 4.4 (Luna's Slice Theorem). *Let G be a reductive algebraic group acting on an affine variety X and let $x \in X$ be such that the orbit Gx is closed. Then, there exists an étale slice S at x .*

We remark that, if X is smooth at x then we automatically have that S is smooth at x and that $T_x X = T_x(G.x) \oplus T_x S$. Even more is true.

Theorem 4.5 (Luna's Slice Theorem for smooth varieties). *Under the assumptions of Theorem 4.4, assume moreover that X is smooth at x . Then, there is an étale G_x -invariant morphism $\phi : S \rightarrow T_x S$ such that $\phi(x) = 0, T\phi_x = \text{id}$ and:*

- (a) *The image of ϕ is an open subset U of $T_x S$ satisfying $U = \pi^{-1}(\pi(U))$, where $\pi : T_x S \rightarrow (T_x S)/G_x$ is the projection.*
- (b) *The map $\phi : S \rightarrow T_x S$ is a strongly étale G_x -morphism.*

Let us give a brief sketch on how to construct S in the smooth case. First, we have the following result.

Lemma 4.6. *Let G be a reductive group acting on an affine variety X . Let $x \in X$ be a smooth point, and assume that G_x is reductive. Then, there is a morphism $\phi : X \rightarrow T_x X$ such that:*

- (1) *ϕ is G_x -invariant.*
- (2) *ϕ is étale at x .*
- (3) *$\phi(x) = 0$.*

Proof. Let $\mathfrak{m}_x \subseteq \mathbb{C}[X]$ be the maximal ideal corresponding to $x \in X$, and denote by $d : \mathfrak{m}_x \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2 = (T_x X)^*$ the projection. Note that this map is G_x -equivariant. Since G_x is reductive, we may find a G_x -invariant subspace $W \subseteq \mathfrak{m}_x$ such that the restriction $d|_W : W \rightarrow (T_x X)^*$ is an isomorphism. We denote $\alpha := (d|_W)^{-1}$, which extends to a map between the symmetric algebras: $S(\alpha) : S(T_x^* X) \rightarrow S(W)$. Composing this with the canonical map $S(W) \rightarrow \mathfrak{m}_x \subseteq \mathbb{C}[X]$, we get a map $\phi : X \rightarrow T_x X$. It is easy to check that ϕ satisfies (1), (2) and (3). \square

Now, assume that $x \in X$ is smooth, and $G.x$ is closed. Since G_x is reductive, we may find a G_x -splitting: $T_x X = T_x G.x \oplus N$. Now let ϕ be the map constructed in Lemma 4.6 and let $U := \phi^{-1}(N)$. We remark, first, that U is a G_x -invariant subvariety with $x \in U$ and U is smooth at x . Moreover, $G \times_{G_x} U$ is smooth at $(1, x)$ and $G \times_{G_x} U \rightarrow X$ is étale at $(1, x)$, this follows because $T_{(1,x)}(G \times_{G_x} U) = (T_1 G \oplus T_x U)/T_1 G_x$. A theorem now is that there exists an affine open subset $S \subseteq U$ that is a required slice. For details, the reader may consult [D].

4.3. Applications of the slice theorem. Let us see a few consequences of Theorems 4.4, 4.5 in the study of the geometry of X and $X//G$.

Proposition 4.7. *Let G act on an affine variety X . Assume that the stabilizer of any point of X is trivial. Then, the map $\pi : X \rightarrow X//G$ is a principal G -bundle.*

Proof. We remark, first, that if the stabilizer of any point is trivial then any orbit is closed (otherwise the closure of an orbit will contain a point with nontrivial stabilizer, see Corollary 2.3.) Now, for any $x \in X//G$, choose a lifting $\bar{x} \in X$, and let S be an étale slice through \bar{x} . Since $G_{\bar{x}} = \{1\}$, $G \times_{G_{\bar{x}}} S = G \times S$, and we have a Cartesian diagram

$$\begin{array}{ccc} G \times S & \xrightarrow{\psi_{S,\bar{x}}} & X \\ \downarrow \pi & & \downarrow \pi \\ S & \xrightarrow{\overline{\psi_{S,\bar{x}}}} & X//G \end{array}$$

With the maps $\psi_{S,\bar{x}}$ and $\overline{\psi_{S,\bar{x}}}$ being étale. This is precisely the definition of a principal G -bundle. \square

Proposition 4.8. *Let G act on the affine, smooth variety X . Assume that the stabilizer of any point of X is trivial. Then, $X//G$ is smooth.*

Proof. The only difference with the previous proposition is that, thanks to Theorem 4.5 we can take S to be smooth at \bar{x} . Since $\overline{\psi_{S,\bar{x}}}$ is étale, this shows that $X//G$ is smooth at x . \square

Proposition 4.9. *Let G act on an affine variety X , and let $x \in X$ be a point such that $G.x$ is closed. Then, there exists an open neighborhood U of x such that for every $y \in U$ there exists $g \in G$ with $g^{-1}G_y g \subseteq G_x$.*

Proof. Let S be an étale slice through X , and let $U = \text{im } \psi_{S,x}$. Since we have a Cartesian diagram

$$\begin{array}{ccc} G \times_{G_x} S & \xrightarrow{\psi_{S,\bar{x}}} & X \\ \downarrow \pi & & \downarrow \pi \\ S//G_x & \xrightarrow{\overline{\psi_{S,\bar{x}}}} & X//G \end{array}$$

we see that for every $y \in U$, $G_y = G_{y'}$ for any $y' \in \psi_{S,x}^{-1}(y)$. Now, for $(g, s) \in G \times S$, the stabilizer of its image in $G \times_{G_x} S$ is precisely $g(G_x)_s g^{-1}$. Thus, if $y = gs$, $g^{-1}G_y g \subseteq (G_x)_s \subseteq G_x$. \square

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