Inv. th'y Lec 10.

Chevalley restr'n thm & geometric quotients

1) Chevalley restr'n thm
2) Torus actions

We want to describe $\mathbb{C}[\mathfrak{g}]^G$. Let

1) Let G be s/simple alg. gr'p, $\mathfrak{g} = Lie(G)$. $\mathfrak{t} \subset \mathfrak{g}$ Cartan, W - Weyl group, $W \hookrightarrow GL(\mathfrak{t})$. Recall that W is a finite group gen'd by the reflections s_α, $\alpha \in R$ (root system), in part'r W is a complex refl'n group $\Rightarrow \mathbb{C}[\mathfrak{t}]^W$ is the polyn'l alg'a in $\dim \mathfrak{t}$ variables

On the other hand $W = N_G(\mathfrak{t})/T$. Consider the restr'n map $\mathbb{C}[\mathfrak{g}] \xrightarrow{r} \mathbb{C}[\mathfrak{t}]$. The restriction of a G-invariant element is $N_G(\mathfrak{t})$ invariant so r maps $\mathbb{C}[\mathfrak{g}]^G$ to $\mathbb{C}[\mathfrak{t}]^W$. Note that r preserves natural gradings

Thm 1 (Chevalley) $r: \mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}]^W$. In part'r $\mathbb{C}[\mathfrak{g}]^G$ is the alg'a of polynomials in $\dim \mathfrak{t}$ variables

Proof: The strategy is as follows:

(i) Show $r^*: \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{t}]^W \rightsquigarrow$ domin $\psi: \mathfrak{t}/W \to \mathfrak{g}/\!/G$.

(ii) Prove ψ is a bijection between dense subsets corresp to "reg'r el'ts" $\Rightarrow$ birat

(iii) Prove ψ is finite by checking that $\psi^{-1}(0) = \{0\}$ (recall ψ is $\mathbb{C}^\times$-equiv't).

Now $\mathfrak{g}/\!/G$ is normal so ψ is finite & birational $\Longrightarrow \psi$ is iso

(i) We need to show $G\mathfrak{t}$ is dense in $\mathfrak{g}$. We say that $x \in \mathfrak{g}$ is <u>regular</u> if $\mathfrak{z}_\mathfrak{g}(x)$ is conj'te to $\mathfrak{t}$. Since $x \in \mathfrak{z}_\mathfrak{g}(x)$ any regular x is s/simple. Let $\mathfrak{g}^{reg}$ be the subset of regular el'ts in $\mathfrak{g}$ & $\mathfrak{t}^{reg} = \mathfrak{g}^{reg} \cap \mathfrak{t}$. Note that $\mathfrak{t}^{reg} = \{x \in \mathfrak{t} \mid \langle \alpha, x \rangle \neq 0 \ \forall \alpha \in R\}$: if $\langle \alpha, x \rangle = 0 \Longrightarrow \mathfrak{g}_\alpha = Span(e_\alpha, h_\alpha, f_\alpha) \subset \mathfrak{z}_\mathfrak{g}(x)$, it's not conj'te to a subalg'a in $\mathfrak{t}$. Note that $\mathfrak{g}^{reg} = G\mathfrak{t}^{reg}$

It's enough to prove $\mathfrak{g}^{reg} \subset \mathfrak{g}$ open

Consider the action morphism $G \times \mathfrak{t}^{reg} \to \mathfrak{g}$, $a(g,x) \mapsto Ad(g)x$; $\mathfrak{g}^{reg} = \operatorname{im} a$

It's enough to show: $d_{(g,x)} a$ is surj'ive $\forall (g,x)$. a is G-equiv't so it's enough
11 to consider $g=1$. Here $d_{(1,x)} a(y,z) = [y,x] + z$. But $\operatorname{im} ad(x) \supset \mathfrak{g}_\alpha \ \forall \alpha \in R$,

$[x,\cdot]$ acts on $\mathfrak{g}_\alpha$ by $\langle\alpha, x\rangle \neq 0$. So $\operatorname{im} d_{(1,x)}a \supset \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha \oplus \mathfrak{t} = \mathfrak{g}$. So we see that $\mathfrak{g}^{reg} \subset \mathfrak{g}$ is open & $\psi: \mathfrak{t}/W \to \mathfrak{g}/\!/G$ is dominant

(ii) Now note that we have a commutative diagram:

$$\begin{tikzcd} \mathfrak{t} \arrow[r, hook] \arrow[d, "\pi_W"'] & \mathfrak{g} \arrow[d, "\pi_G"] \\ \mathfrak{t}/W \arrow[r, "\psi"'] & \mathfrak{g}/\!/G \end{tikzcd}$$

So for $z \in \mathfrak{g}/\!/G$, we have $\psi^{-1}(z) = (\pi_G^{-1}(z) \cap \mathfrak{t})/W$. Note that $\pi_G(\mathfrak{g}^{reg})$ is dense in $\mathfrak{g}/\!/G$. Pick $x \in \mathfrak{g}^{reg}$, $z := \pi_G(x)$, x is s/simple $\Rightarrow Gx$ is the unique closed orbit in $\pi^{-1}(z)$ $\Leftrightarrow$ unique s/simple orbit. But $\pi^{-1}(z) \cap \mathfrak{t}$ consists of s/simple el-ts $\Rightarrow \pi^{-1}(z) \cap \mathfrak{t} = Gx \cap \mathfrak{t}$. It remains to show $Gx \cap \mathfrak{t}$ is a single W-orbit. Let $x_1, x_2 \in Gx \cap \mathfrak{t}$. Then $x_1, x_2 \in \mathfrak{t}^{reg} \Leftrightarrow \mathfrak{z}_\mathfrak{g}(x_1) = \mathfrak{z}_\mathfrak{g}(x_2) = \mathfrak{t}$; $\exists\, g \in G$ s.t. $gx_1 = x_2 \Rightarrow g\,\mathfrak{z}_\mathfrak{g}(x_1) = \mathfrak{z}_\mathfrak{g}(x_2) \Leftrightarrow g \in N_G(\mathfrak{t})$ $\Rightarrow Wx_1 = Wx_2$. So indeed $\psi^{-1}(z) = \pi_W(x_i)$ is a single pt.

(iii) reduces to check $\pi_G^{-1}(0) \cap \mathfrak{t} = \{0\}$. But $\pi^{-1}(0)$ consists of nilp el'ts & $\mathfrak{t}$ consists of s/simple el'ts hence our claim $\square$

Rem. $\cdot$ From $\mathfrak{g}/\!/G \xrightarrow{\sim} \mathfrak{t}/W$ & the claim that every fiber of π_G contain a unique closed ($\Leftrightarrow$ s/simple) orbit, we see that s/simple G-orbits in $\mathfrak{g}$ are classified by the W-orbits in $\mathfrak{t}$

$\cdot$ For $\mathfrak{g} = \mathfrak{sl}_n$, the Thm reads that $\mathbb{C}[\mathfrak{g}]^G$ is polyn'l alg'a in $F_i(A) = \operatorname{Tr}(A^i)$, $i = 2, \ldots n$, which was basically mentioned in Lec 1.

2) Here we consider actions of $T = (\mathbb{C}^\times)^m$ on vector space $V = \mathbb{C}^n$ (linear). The action is diagonalizable in some basis $v_1, \ldots v_n \in V$ and given by $t.v_i = \chi_i(t)v_i$ for characters χ_i. We are going to describe the algebra of invariants $\mathbb{C}[V]^T$ and the closure $\overline{Tv}$ for $v \in V$. The latter is the first step in the theory of toric varieties and also is used in the Hilbert-Mumford thm, which is our next topic

2.1) Algebra $\mathbb{C}[V]^T$. Let $x_1 \ldots x_n \in V^*$ be the dual basis to $v_1 \ldots v_n$. Then $t.\, x_1^{\alpha_1} \ldots x_n^{\alpha_n} = \chi_1(t)^{-\alpha_1} \ldots \chi_n(t)^{-\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. So $\mathbb{C}[V]^T$ is spanned by monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ w $\sum_{i=1}^n \alpha_i \chi_i = 0$ (we use additive notation for $\mathfrak{X}(T)$) The description of algebra structure on $\mathbb{C}[V]^T$ in a HW problem

Rem: For $I \subset \{1..n\}$ consider open subset $V_I = \{v \in V \mid v_i \neq 0 \ \forall i \in I\}$
Then $\mathbb{C}[V_I] = \mathbb{C}[V][x_i^{-1}, i \in I]$ & $\mathbb{C}[V_I]^T$ is spanned by monomials $x_1^{d_1} \dots x_n^{d_n}$ w. $d_i \in \mathbb{Z}$ for $i \in I$, $d_i \in \mathbb{Z}_{\geq 0}$ for $i \notin I$ & $\sum_{i=1}^n d_i \chi_i = 0$

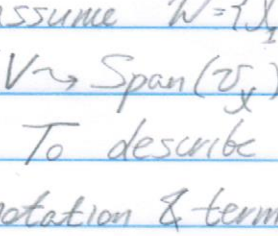
2.2) Orbit closures. Pick $v \in V$, $v = \sum_{\chi \in \mathcal{W}} v_\chi$, where $\mathcal{W} \subset \{\chi_1 \dots \chi_n\}$ & v_χ a wt. vector w. character χ, so $t.v = \sum_{\chi \in \mathcal{W}} \chi(t) v_\chi$. In part'r $Tv \subset Span(v_\chi) \Rightarrow \overline{Tv} \subset Span(v_\chi)$ So in the study of $\overline{Tv}$ we can assume $\mathcal{W} = \{\chi_1 \dots \chi_n\}$ & all character $\chi_1 \dots \chi_n$ are distinct, $v = \sum_{i=1}^n v_i$
($V \rightsquigarrow Span(v_\chi)$)

To describe the orbit closure we need some preparation, mostly notation & terminology.

By the weight polytope, P we mean the convex hull of $\mathcal{W}$.(*) In particular we can talk about faces of P, the largest one is P itself. We also formally adjoin the empty face $\varnothing$, it's contained in any other face.

(*) It's a rational convex polytope in $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$

Ex: $T = (\mathbb{C}^\times)^2$, $\mathcal{W} = \{e_1, 2e_1, 3e_1, 3e_1 + 2e_2, e_1 + e_2, e_2\}$. P has 10 faces: $\varnothing$, 4 0-dim'l, 4 1-dim'l and 1 2-dim'l, which is P



Def: A face F of P is called admissible, if $F = P$ or $\exists$ a hyperplane Γ passing through 0 so that $F = \Gamma \cap P$ (and P lies on one side of Γ)

In our example, the admissible faces are $\varnothing$, $\{e_2\}$, the interval $[e_1, 3e_1]$, P.

Ex': Let $0 \in P$. Then a face F is admissible $\Leftrightarrow 0 \in F$.

Note that there is at most one face F of P such that $0 \in \mathring{F}$ (the relative interior of F). It exists when $0 \in P$, in which case we write F_0 for the face. If $0 \notin P$, set $F_0 := \varnothing$.

Finally, for an admissible face F, set $v_F := \sum_{\chi \in F \cap \mathcal{W}} v_\chi$. In the example above, these vectors are v, $v_{e_1} + v_{2e_1} + v_{3e_1}$, v_{e_2}, 0

Thm 2: There is a one-to-one correspondence between the T-orbits in $\overline{Tv}$ and admissible faces: the orbit corresp to F is Tv_F

In particular, the closed orbit is Tv_{F_0} (and $0 \in \overline{Tv} \Leftrightarrow F_0 = \phi$, $\overline{Tv} = Tv \Leftrightarrow 0 \in \mathring{P}$)

The proof is in 3 steps: (i) $Tv_F \subset \overline{Tv}$ $\forall$ admissible F

(ii) If $v_1 \in \overline{Tv}$, then $v_1 = \sum_{\chi \in W \cap F} z_\chi v_\chi$, $z_\chi \in \mathbb{C}\setminus\{0\}$ for admissible F (uniquely determined from v_1)

(iii) $Tv_1 = Tv_F$

In (i) we need to define a limit under the action of $\mathbb{C}^\times$ as $t \to 0$. Let X be a (separated) variety w. $\mathbb{C}^\times$-action. For $x \in X$, we have a map $\mathbb{A}^1\setminus\{0\} \to X$ given by $t \mapsto t.x$. If it extends to $\mathbb{A}^1$, this extension is unique. The image of 0 under the extension is denoted by $\lim_{t\to 0} tx$ (it's the same as the limit in the usual topology). For $X = U$, a lin'r rep'n, for $x = \sum_i u_i$ (w. $t.u_i = t^i u_i$), $\lim_{t\to 0} t.x$ exists $\Leftrightarrow u_i = 0, i<0$, in which case $\lim_{t \to 0} t.x = u_0$. For arbitrary affine X, we $\mathbb{C}^\times$-equiv'ly embed it into some U, and can compute lim from there

Proof of Thm 2: (i) By def'n of admissible face we have $\nu \in Hom(\mathbb{C}^\times, T) = \mathfrak{X}(T)^*$ s.t $\nu|_P \geq 0$, $\nu|_F = 0$. By the previous comp'n of limit, $\lim_{t\to 0} \nu(t)v = v_F$. ($\langle \nu, \chi \rangle \geq 0$ $\forall \chi \in W$ & $\langle \nu, \chi \rangle = 0 \Leftrightarrow \chi \in F \cap W$)

(ii) Let $v_1 \in \overline{Tv}$, $v_1 = \sum_\chi z_\chi v_\chi$ w. $z_\chi \neq 0$ for $\chi \in I \subset W$. Let $\tilde{I}$ consist of all $\chi' \in W$ s.t $\exists$ $n_\chi \in \mathbb{Z}_{\geq 0}$ for $\chi \in W$, $n_{\chi'} > 0$ s.t $\sum_\chi n_\chi \chi \in Span_{\mathbb{Z}}(I)$.

Exercise: $Conv(\tilde{I})$ is an admissible face

We claim $I = \tilde{I}$. Clearly $I \subset \tilde{I}$. If $\chi' \in \tilde{I}$, then $\exists$ $d_\psi, \psi \in I$, s.t $\sum_\chi n_\chi \chi = \sum_{\psi \in I} d_\psi \psi$. The monomial $\prod_\chi x_\chi^{n_\chi} \cdot \prod_\psi x_\psi^{-d_\psi} \in \mathbb{C}[V_I]$ is inv't. It's 1 on $v \in V_I$ and $Tv_1 \subset \overline{Tv}$ (in V_I) so it's 1 on v_1. So $z_{\chi'} \neq 0 \Rightarrow \chi' \in I$. This proves (ii).

(iii) Let $V(F) = Span(v_\chi \mid \chi \in F \cap W)$, $V^0(F) = \overbrace{\{\sum_\chi z_\chi v_\chi \mid z_\chi \neq 0 \ \forall \chi \in F \cap W}^{:= V(F) \cap V_{F \cap W}}$ (e.g in example, for $F = [e_1, 3e_1]$, $V(F) = Span(v_{e_1}, v_{2e_1}, v_{3e_1})$) We have $Tv_1, Tv_F \subset V^0(F)$. All orbits in $V^0(F)$ have dim = dim F $\Rightarrow$ closed $\Rightarrow$ separated by invariants. But all inv't monomials on $V^0(F)$ extend to invariant

$$= \{ \sum_i z_i v_i \mid z_i \neq 0 \text{ for } \chi_i \in F \cap \mathcal{W} \}$$

monomials on $V_{F\cap \mathcal{W}}$. Since $v_F \subset \overline{Tv}$, for any such monomial f we have $f(v) = f(v_F)$. But for similar reason, $f(v_1) = f(v)$. So $f(v_1) = f(v_F)$ $\forall$ inv't monomial on $V^0(F) \Rightarrow \forall f \in \mathbb{C}[V^0(F)]^T$. Since all orbits are closed, they are separated by invariants $\Rightarrow Tv_1 = Tv_F$ $\square$

Cor (of proof of (i)) X affine, $x \in X$, $y \in X$ s.t. $Ty \subset \overline{Tx}$. $\exists\ \gamma : \mathbb{C}^\times \to T$ s.t. $\lim_{t\to 0} \gamma(t)x \in Ty$.

Rem: One common feature of $G \curvearrowright \mathfrak{g}$, $T \curvearrowright V$, is that closure of any orbit contains only finitely many orbits (for $x = x_s + x_n \in \mathfrak{g}$, $\overline{Gx}$ consists of orbits of el'ts $x_s + x_n'$ w. $x_n' \in \overline{Z_G(x_s)^\circ x_n}$, and then we can use finiteness of the number of nilp. orbits.

This is not the case in general, an example is provided by left action $GL_n(\mathbb{C}) \curvearrowright Mat_n(\mathbb{C})$ – w. open orbit given by non-degenerate matrices and infinitely many orbits, e.g. of the form $(z_1 v, z_2 v, \dots, z_n v)$ $z_i \in \mathbb{C}$. Here $n > 1$.