HW4 Solutions.

P1: For $A \in Mat_{n,k}(\mathbb{C})$, we have $A^TA \in \mathcal{D}_n^+$. Also for $g \in O_n$, have $(gA)^T gA = A^T g^T g A = [g^Tg = id] = A^TA$. So the map $\psi: A \mapsto A^TA: Mat_{n,k}(\mathbb{C}) \to \mathcal{D}_n^+$ is O_n-inv't. Since $\mathcal{D}_n^+$ is normal, what we need to check, thx to Igusa's criterium, is that ψ is surjective & every fiber contains a unique closed orbit.

- ψ is surjective: Note that for $B \in GL_k$, we have $\psi(AB) = B^T \psi(A) B$. Every $C \in \mathcal{D}_n^+$ is GL_k-conjugate to a matrix of the form $diag(1,..1,0,..0) = \psi\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 & \\ & & & 0\end{pmatrix}$. This shows that ψ is surjective
- As in the proof of the fund. thm for GL_n the proof of the claim that every fiber of ψ contains a single closed orbit reduces to the following two claims: (i) if GA ($G = O_n$) is closed, then $im A \oplus \ker A^T = \mathbb{C}^n$ (which is equiv't to $(\cdot,\cdot)_{im A}$ is non-degenerate)

(ii) $\forall\, C \in \mathcal{D}_n^+$, $\psi^{-1}(C)$ contains a unique orbit GA w $(\cdot,\cdot)_{im A}$ non-deg'te.

Proof of (i): let $U_0 = (im A) \cap (im A)^\perp \neq 0$. Pick a compl't U_2 to U_0 in U & U_1 to U_0 in $U_0^\perp$. Define a one-parameter subgroup $\gamma: \mathbb{C}^\times \to O_n$ by $\gamma(t)|_{U_0} = t \cdot id$, $\gamma(t)|_{U_1} = id$, $\gamma(t)|_{U_2} = t^{-1} id$ ~~where we assume that the form is s.t.~~, the image of γ is indeed in O_n and $\lim_{t\to 0} \gamma(t) A$ is the comp'n $pr_{U_1} \circ A \notin A$. This finishes (i)

Proof of (ii) Let A, A' satisfy $A^TA = A'^TA' = C (\in \mathcal{D}_n^+)$ & $im A, im A' \subset \mathbb{C}^n$ are ~~orthog~~ non-degenerate w.r.t. $(\cdot,\cdot)$ Since $im A \oplus \ker A^T = \mathbb{C}^n$, we have $rk\, A = rk\, C$, similarly $rk\, A' = rk\, C$. It follows that $im A$, $im A'$ are conjugate in O_n. We can assume that $im A = im A' =: U$. Now let's view A as a collection of column vectors $v_1, .. v_k \in U$, similarly get vectors $v_1', .. v_k' \in U$, it's enough to show $\exists\, g \in O(U)$ w. $gv_i = v_i'$. Note that C is the Gram matrix for $v_1 .. v_k$ (and for $v_1', .. v_k'$). We have a $\dim U$-element subset $I \subset \{1,..k\}$ s.t. the corresponding minor in C

is nonzero $\Leftrightarrow$ v_i, $i\in I$, & $v_i' \in I$, are bases of U. Their Gram matrices are the same so $g\in GL(U)$ def'd by $gv_i = v_i'$ is, actually, in $O(U)$. Since v_j, $j\notin I$ is uniquely recovered from v_i $i\in I$, and C (and same for v_j''s) we see that $gv_j = v_j'$. This finishes the proof.

P2 It's enough to show that $\mathbb{C}[Mat_n]^{O_n}$ is gen'd by traces of monomials in A, A^T. From here and $\mathbb{C}[Mat_n]^{O_n} \twoheadrightarrow \mathbb{C}[Mat_n^+]^{O_n}$ it follows that $\mathbb{C}[Mat_n^+]^{O_n}$ is gen'd by $Tr(A^i)$, $i>0$ (on Mat_n^+, have $A=A^T$). By the proof of a Thm in the GL_n-case, $Tr(A^i)$ for $i>n$ lies in the subalgebra gen'd by $Tr(A^j)$, $j=1..n$. These el'ts are alg. indep. to see this we can restrict them to diagonal matrices, as in the lecture.

To prove that $\mathbb{C}[Mat_n]^{O_n}$ is gen'd by ~~monomials in~~ traces of monomials in A, A^T, it's enough to check that $(Mat_n^{\otimes k})^{*O_n}$ is spanned by products of traces in B_i's, $i=1..k$. Thanks to the presence of a non-degenerate form on U, we have $Mat_n(\mathbb{C}) = U\otimes U^* \xrightarrow{\sim} U\otimes U$. The permutation of factors map $U\otimes U^* \to U^*\otimes U$ corresponds, on the level of matrices, to $A\mapsto A^*$. In our case the permutation map $U\otimes U \to U\otimes U$ is, therefore, $A\mapsto A^T$.

Now identify $Mat_n^{\otimes k}$ w $U^{\otimes 2k}$ in a nat'l way. We already know that $(U^{\otimes 2k})^{*,O_n}$ is gen'd by elements F_I, where $F_I(u_1\otimes .. \otimes u_{2k}) = \prod_{\alpha_1<\alpha_2} (u_{\alpha_1}, u_{\alpha_2})$, and I is a partit'n of $\{1,..2k\}$, ~~and the~~ into pairs and the product is taken over all pairs $(\alpha_1<\alpha_2)$ in I. We claim that F_I gives a required product of traces. For this it's enough to understand an effect of applying a single pairing: $u_1\otimes .. \otimes u_{2k} \mapsto (u_{\alpha_1}, u_{\alpha_2})\, u_1\otimes ... \otimes \hat{u}_{\alpha_1}\otimes .. \otimes \hat{u}_{\alpha_2}\otimes .. \otimes u_{2k}$ on $B_1\otimes .. \otimes B_k$. Since we can permute the matrices, it's enough to understand the situation we $\alpha_1, \alpha_2 \in \{1,2,3,4\}$. We can also assume $\alpha_1\in\{1,2\}$

Case 1: $\alpha_1=1, \alpha_2=2$. We get $B_1\otimes .. \otimes B_k \to Tr(B_1) B_2\otimes .. \otimes B_k$.

Case 2: $\alpha_1=2, \alpha_2=3$.; That's the map $U\otimes U^*\otimes U\otimes U^*$ contracting 2nd & 3rd factor. This is the matrix multiplication: $B_1\otimes B_2 \mapsto B_1 B_2$.

Case 2': $\alpha_1=1, \alpha_2=4$: $B_1 \otimes B_2 \mapsto B_2 B_1$

Case 3: $\alpha_1=1, \alpha_2=3$: Thx to $u_1 \otimes u_2 \mapsto u_2 \otimes u_1$, corresp. to $B_1 \mapsto B_1^T$ we reduce to Case 2 and the map is $B_1 \otimes B_2 \mapsto B_1^T B_2$.

Case 3': $\alpha_1=2, \alpha_2=4$: similarly to the previous case, we get $B_1 \otimes B_2 \mapsto B_1 B_2^T$.

To summarize: applying a single pairing amounts to either taking trace or multiplying matrices (or their transposes). This finishes the proof of an interpretation of F_I on the level of matrices.

<u>P3</u>: For U we take the group of <u>lower-triangular</u> matrices.

Note that $f: \Lambda^k \mathbb{C}^n \to \mathbb{C}$ is indeed U-invariant. Let $X = C\,Gr(k,n)$. We claim that $X_f = U\,\mathbb{C}^\times (e_1 \wedge \dots \wedge e_k)$. Indeed, X_f consists of all $v_1 \wedge \dots \wedge v_k \neq 0$ s.t. $\operatorname{Span}(v_1 \dots v_k) \cap \operatorname{Span}(e_{k+1} \dots e_n) = 0$. Applying a uni-modular transformation to $v_1 \dots v_k$, can assume that $v_1 = \alpha e_1 + \sum_{j=k+1}^{n} a_{1j} e_j$; $v_i = e_i + \sum_{j=k+1}^{n} a_{ij} e_j$, $i=2, \dots k$. Clearly $\exists\, g \in U$ s.t. $v_1 = g\alpha e_1$, $v_i = g e_i$, $i=2 \dots n \Rightarrow$ $v_1 \wedge \dots \wedge v_k = g(\alpha(e_1 \wedge \dots \wedge e_k))$, which proves our statement. So $\mathbb{C}[X_f]^U \hookrightarrow \mathbb{C}[f^{\pm 1}] \rightsquigarrow \mathbb{C}[X_f]^U \hookrightarrow \mathbb{C}[f^{\pm 1}]$. Also $\mathbb{C}[f] \subset \mathbb{C}[X]^U \subset \mathbb{C}[f^{\pm 1}]$. Assume $\mathbb{C}[X]^U \neq \mathbb{C}[f]$. Then $f^{-1} \in \mathbb{C}[X]^U$, which is nonsense: f is clearly non-invertible ($f(0)=0$).

P4: a) Note that $(\mathbb{C}[G/U]\otimes B)^T$ is fin. gen'd graded G-algebra & $[(\mathbb{C}[G/U]\otimes B)^T]^U = (\mathbb{C}[G/U]^U \otimes B)^T = (\mathbb{C}[\mathfrak{X}^+]\otimes B)^T = B$. So ~~every~~ it remains to check that every graded G-algebra has the form $(\mathbb{C}[G/U]\otimes B)^T$ for a graded algebra B. Note that $V(\lambda+\mu)$ occurs in $V(\lambda)\otimes V(\mu)$ w. mult $=1$ and this copy of $V(\lambda+\mu)$ is gen'd by $v_\lambda \otimes v_\mu$. It follows that the product $A_\lambda \otimes A_\mu \to A_{\lambda+\mu}$ is completely recovered from its restriction to $A_\lambda^U \otimes A_\mu^U \to A_{\lambda+\mu}^U$. Now note that such restrictions for A and for $(\mathbb{C}[G/U]\otimes A^U)^T$ coincide. This establishes a G-equiv. iso $A \xrightarrow{\sim} (\mathbb{C}[G/U]\otimes A^U)^T$.

b) The claim that $A_{\leq i}$ is an algebra filtration follows from the observation that all highest weights that appear in $V(\lambda)\otimes V(\mu)$ are $\leq \lambda+\mu$. Note that $A^U \simeq (\operatorname{gr} A)^U$ so $(\operatorname{gr} A)^U$ is fin. gen'd $\Rightarrow \operatorname{gr} A$ is fin. gen'd.