CRYSTALS

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In these notes we introduce the crystal structures of modules over Kac-Moody algebras obtained from Berenstein-Kazhdan perfect bases, especially on the complexified Grothendieck groups of type A Kac-Moody categorifications.

In Section 1 we describe the structure of simple objects in an $\mathfrak{sl}_2$-categorification. In Section 2, we introduce the Berenstein-Kazhdan perfect bases of integrable highest weight representations of a Kac-Moody algebra. Finally in Section 3, we apply what we have in the first two sections to the example of categorical $\widehat{\mathfrak{sl}_l}$-action on modules over cyclotomic Hecke algebras, and conclude that this is a categorification of an irreducible $\widehat{\mathfrak{sl}_l}$-module.

1. Simple objects in an $\mathfrak{sl}_2$-categorification

1.1. **Reminder and notation.** Let $\mathcal{C}$ be a general artinian and noetherian $\mathbb{F}$-linear abelian category equipped with a categorical $\mathfrak{sl}_2$-action given by the endofunctors E and F, the parameter $q \in \mathbb{F}^\times$ and $a \in \mathbb{F}$, where $a \neq 0$ if $q \neq 1$, and $L \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$. We adopt some notation from [Si] and [CR]:

- Let $[\mathcal{C}] = K_0(\mathcal{C}) \otimes \mathbb{C}$ denote the complexified Grothendieck group of $\mathcal{C}$ and $\mathcal{H}_q^{\mathrm{aff}}(n)$ denote the affine Hecke algebra generated by $X_1, \cdots, X_n, T_1, \cdots, T_{n-1}$ subject to the Hecke relations.
- For some $U \in \mathcal{C}$, denote $h_+(U) := \max\{j : E^jU \neq 0\}$, $h_-(U) := \max\{j : F^jU \neq 0\}$, and $d(U) := h_+(U) + h_-(U) + 1$. Also, denote the *socle* of U by $\mathrm{soc}(U)$, which is the maximal semisimple subobject of U in $\mathcal{C}$, and the *head* by $\mathrm{head}(U)$, which is the maximal semisimple quotient.
- $E^{(i)}$, $F^{(i)}$ denote the categorified divided powers.
- Let $\mathfrak{m}_n \subseteq P_n := \mathbb{F}[X_1^{\pm}, \dots, X_n^{\pm}]$ be the ideal generated by $(X_i - a)$, $i = 1, \dots, n$. Let $\mathfrak{n}_n := \mathfrak{m}_n^{\mathfrak{S}_n} \subseteq \mathcal{H}_q^{\mathrm{aff}}(n)$. Let $\mathcal{N}_n$ be the category of $\mathcal{H}_q^{\mathrm{aff}}(n)$-modules with locally nilpotent $\mathfrak{n}_n$-action. Since $\mathfrak{n}_n$ is contained in the center of $\mathcal{H}_q^{\mathrm{aff}}(n)$, the quotient $\overline{\mathcal{H}(n)} = \mathcal{H}_q^{\mathrm{aff}}(n)/\mathfrak{n}_n\mathcal{H}_q^{\mathrm{aff}}(n)$ is an algebra. For $0 \leq i \leq n$, denote by $B_{i,n}$ the image of the subalgebra $\mathcal{H}_q^{\mathrm{aff}}(i)$ inside $\overline{\mathcal{H}(n)}$. Define the Kato modules $K_n := \mathcal{H}_q^{\mathrm{aff}}(n) \otimes_{P_n} P_n/\mathfrak{m}_n \cong (\mathcal{H}_q^{\mathrm{aff}}(n)/\mathfrak{n}_n)c_n^\tau$ to be the unique simple module in $\mathcal{N}_n$, where $c_n^\tau = \sum_{w \in \mathfrak{S}_n} q^{-\ell(w)}\tau(T_w)T_w$ for $\tau \in \{\mathrm{triv}, \mathrm{sign}\}$.
- As in [Si, Proposition 3.3], for any $U \in \mathcal{C}$ and $n > 0$, $E^n(U)$ has a natural left $\mathcal{H}_q^{\mathrm{aff}}(n)$-module structure. It induces a morphism $\gamma_n : \mathcal{H}_q^{\mathrm{aff}}(n) \to \mathrm{End}(E^n)$ defined by $T_i \mapsto \mathbf{1}_{E^{n-i-1}}T\mathbf{1}_{E^{i-1}}$ and $X_j \mapsto \mathbf{1}_{E^{n-i}}L\mathbf{1}_{E^{i-1}}$.
- Given $d \geq 0$, let $\mathcal{C}^{\leq d}$ be the full Serre subcategory of $\mathcal{C}$ consisting of all simple objects S such that $d(S) \leq d$. Let $[\mathcal{C}]^{\leq d}$ be the maximal submodule of $[\mathcal{C}]$ containing all modules of dimension $\leq d$. Clearly $[\mathcal{C}^{\leq d}] \subset [\mathcal{C}]^{\leq d}$. In fact this is an equality.

1.2. **Simples in $\mathcal{C}$.** In this subsection, we focus on the categorical action of E and F on a simple object S in $\mathcal{C}$. In general, ES and FS (or more generally, $E^{(i)}S$ and $F^{(i)}S$) are not necessarily simple, but their socles and heads are. Also we prove some results describing $\mathrm{End}(E^{(i)}S)$.

The following result is due to Chuang-Rouquier [CR, Prposition 5.20].

Proposition 1.1. *Let S be a simple object of $\mathcal{C}$, and let $n = h_+(S)$. Then, for every $i \leq n$:*

(a) *The object $E^{(n)}S$ is simple.*

(b) *The socle and the head of $E^{(i)}S$ are isomorphic to a simple object S' of $\mathcal{C}$. We have $\mathcal{H}_q^{\mathrm{aff}}(i)$-equivariant $\mathcal{C}$-isomorphisms: $soc(E^iS) \cong head(E^iS) \cong S' \otimes K_i$.*

(c) *The canonical homomorphism $\gamma_i(S) : \mathcal{H}_q^{\mathrm{aff}}(i) \to \mathrm{End}_{\mathcal{C}}(E^iS)$ factors through $B_{i,n}$. Moreover, it induces an isomorphism $B_{i,n} \xrightarrow{\cong} \mathrm{End}_{\mathcal{C}}(E^iS)$.*

$$\begin{tikzcd}
& \mathcal{H}_q^{\mathrm{aff}}(i) \arrow[dl, "can"'] \arrow[d, "\gamma_i(S)"] \\
B_{i,n} \arrow[r, dotted, "\cong"] & \mathrm{End}(E^iS).
\end{tikzcd}$$

(d) *We have $[E^{(i)}(S)] - \binom{n}{i}[S'] \in [\mathcal{C}]^{\leq d(S')-1}$.*

The corresponding statements with E replaced by F and $h_+(S)$ by $h_-(S)$ hold as well.

To prove the proposition, we need the following two lemmas.

Lemma 1.2. *Let M be an object of $\mathcal{C}$. If $d(S) \geq r$ for any simple subobject (resp. quotient) S of M, then $d(S') \geq r$ for any simple subobject (resp. quotient) of EM or FM.*

Proof. By the weight decomposition of $\mathcal{C}$ ([Si, Proposition 3.5]), it is enough to consider the case where M lies in a single weight space. Let T be a simple submodule of EM, by adjunction, $\mathrm{Hom}(FT, M) \cong \mathrm{Hom}(T, EM) \neq 0$. So there exists S being a simple subobject of M that is a composition factor of FT. Hence, $d(T) \geq d(FT) \geq d(S) \geq r$. The proofs for FM and simple quotients are similar. □

For $1 \leq i \leq j \leq n$, denote by $\mathfrak{S}_{[i,j]}$ the symmetric group on $[i,j] = \{i, i+1, \cdots, j\}$. We define similarly $\mathcal{H}_q^{\mathrm{aff}}([i,j])$ and $\overline{\mathcal{H}([i,j])}$ and we put $c^{\tau}_{[i,j]} = \sum_{w \in \mathfrak{S}_{[i,j]}} q^{-\ell(w)} \tau(T_w) T_w$.

Lemma 1.3. *The $\mathcal{H}_q^{\mathrm{aff}}(i)$-module $c^{\tau}_{[i+1,n]} K_n$ has a simple socle and head.*

Proof. See [CR, Lemma 3.6], or [Ven, Theorem 5.10]. □

Proof of Proposition 1.1. The proof is in several steps.

Step 1. (a) holds when $FS = 0$. Since $[E]$, $[F]$ define an $\mathfrak{sl}_2$-action on $[\mathcal{C}]$, $[F^{(n)}E^{(n)}S] = r[S]$ for some $r \in \mathbb{Z}_{>0}$. By adjointness, $\mathrm{Hom}(F^{(n)}E^{(n)}S, S) = \mathrm{Hom}(E^{(n)}S, E^{(n)}S) \neq 0$. So there exists a nonzero homomorphism $F^{(n)}E^{(n)}S \to S$, hence an isomorphism. Then $F^{(n)}E^{(n)}S \cong S$. If $E^{(n)}S$ has at least two composition factors, then by weight consideration, $F^{(n)}E^{(n)}S$ also has at least two composition factors, and thus cannot be simple. So $E^{(n)}S = S'$ must be simple.

Step 2. (a) holds in general. Let L be a simple quotient of $F^{(r)}S$, where $r = h_-(S)$. Note that, by our choice of r, $FL = 0$ so, by Step 1, $E^{(n+r)}L = T$ is simple and $E^{(n)}E^{(r)}L = \binom{n+r}{r} T$. By adjunction, we have that $\mathrm{Hom}(S, E^{(r)}L) \cong \mathrm{Hom}(F^{(r)}S, L) \neq 0$, so S must be a subobject of $E^{(r)}L$. It follows that $E^{(n)}S$ must be a subobject of $\binom{n+r}{r} T$. So $E^{(n)}S = mT$ for some $m > 0$. Clearly, $m = \dim \mathrm{Hom}(E^{(n)}S, T) = \dim \mathrm{Hom}(S, F^{(n)}T)$. But $ET = 0$, so by Step 1 (with E and F swapped) $\mathrm{soc}(F^{(n)}T)$ is simple. Thus, $m = 1$.

Step 3. (b) holds whenever (a) does. Clearly, (b) holds when $i = n$. But let us observe a bit more. We have $E^n S = n! S'$ for some simple module S'. Thus, $E^n S = S' \otimes R$ for some left $\mathcal{H}_q^{\mathrm{aff}}(n)$-module R in $\mathcal{N}_n$. Since $\dim R = n! = \dim K_n$, we must have $R = K_n$.

For $i < n$ we have, using exactness of E and the above paragraph, that $E^{n-i} \mathrm{soc}(E^{(i)}S) \subseteq E^{n-i}E^{(i)}S \cong S'' \otimes K_n c_i^1$. The $\mathcal{H}_q^{\mathrm{aff}}(n-i)$-module $K_n c_i^1$ has a simple head and socle, (Lemma 1.3), so the same is true for $S'' \otimes K_n c_i^{\tau}$ (as a $\mathcal{H}_q^{\mathrm{aff}}(n-i)$-module in $\mathcal{C}$). It follows that $E^{n-i} \mathrm{soc}(E^{(i)}S)$ is indecomposable as a $\mathcal{H}_q^{\mathrm{aff}}(n-i)$-module in $\mathcal{C}$. Now, if S' is a nonzero summand of $\mathrm{soc}(E^{(i)}S)$, then $E^{n-i}S' \neq 0$ (Lemma 1.2). So $\mathrm{soc}(E^{(i)}S)$ has no more than one summand and hence must be simple. We have $\mathrm{soc}(E^i S) \cong S' \otimes R$ for some $\mathcal{H}_q^{\mathrm{aff}}(i)$-module R in $\mathcal{N}_i$. Since $\dim R = i!$, it follows that $R \cong K_i$. $\mathrm{soc}(E^{(i)}S) = S'$. The proof for the head being simple is similar. It remains to show that the head and the socle are isomorphic.

Step 4. Estimating the dimension of $\mathrm{End}(E^i S)$. Since $S' = \mathrm{soc}(E^{(i)}S)$ is simple, the dimension of $\mathrm{Hom}(M, E^{(i)}S)$ is at most the multiplicity of S' in M. Taking $M = E^{(i)}S$, we get that the dimension of $\mathrm{End}(E^{(i)}S)$ is at most the multiplicity of S' in $E^{(i)}S$. Since $E^{(n-i)}S' \neq 0$, we have that the dimension of $\mathrm{End}(E^{(i)}S)$ is at most the number of composition factors of $E^{(n-i)}E^{(i)}S$. But $E^{(n-i)}E^{(i)}S = \binom{n}{i} S''$. Thus, $\dim(\mathrm{End}(E^{(i)}S)) \leq \binom{n}{i}$. Since $E^i S = i! E^{(i)}S$, it follows that $\dim \mathrm{End}(E^i S) \leq (i!)^2 \binom{n}{i} = \dim B_{i,n}$.

Step 5. (c) holds whenever (a) holds. $\ker \gamma_n(S) \subseteq \mathfrak{n}_n \mathcal{H}_q^{\mathrm{aff}}(n)$ since the former is a proper ideal and the latter is a maximal ideal of $\mathcal{H}_q^{\mathrm{aff}}(n)$. For $i < n$, we have that $\ker \gamma_i(S) \subseteq \mathcal{H}_q^{\mathrm{aff}}(i) \cap \ker \gamma_n(S) \subseteq \mathcal{H}_q^{\mathrm{aff}}(i) \cap (\mathfrak{n}_n \mathcal{H}_q^{\mathrm{aff}}(n))$. Then, we have an induced surjective map $\mathrm{im}\gamma_i(S) \to B_{i,n}$. By Step 4 (that was done under the assumption that (a) holds) this must be an isomorphism and $\gamma_i(S)$ must be surjective.

Step 6. (d) holds whenever (a) holds. In Step 4 we also get that the multiplicity of S' as a composition factor of $E^{(i)}(S)$ is $\binom{n}{i}$. If L is a composition factor of $E^{(i)}S$ with $E^{(n-i)}L \neq 0$, then $L \cong S'$. And since the multiplicity of $\mathrm{head}(E^{(i)}S)$ in $E^{(i)}S$ is also $\binom{n}{i}$ and $\mathrm{head}(E^{(i)}S)$ is not killed by $E^{(n-i)}$, $\mathrm{head}(E^{(i)}S) \cong S' \cong \mathrm{soc}(E^{(i)}S)$. Now we also finish the proof of (b) and we are done. □

Take $i = 1$ in the proposition above, we get a map

$$\tilde{e} : \mathrm{Irr}\mathcal{C} \to \mathrm{Irr}\mathcal{C} \sqcup \{0\}, \ S \mapsto \mathrm{soc}(ES) = \mathrm{head}(ES), \tag{1}$$

and similarly

$$\tilde{f} : \mathrm{Irr}\mathcal{C} \to \mathrm{Irr}\mathcal{C} \sqcup \{0\},\ S \mapsto \mathrm{soc}(FS) = \mathrm{head}(FS). \tag{2}$$

Note that if $ES = 0$ then $\tilde{e}(S) = \mathrm{soc}(ES) = 0$; If $\tilde{e}(S) \neq 0$, we have $\tilde{f}\tilde{e}S = S$.

2. Berenstein-Kazhdan perfect bases

In this section we introduce the Berenstein-Kazhdan perfect bases. In a $\mathfrak{g}$-module, a basis is *perfect* in the sense that it behaves nicely under the action of Chevalley generators. It equips the $\mathfrak{g}$-module with a crystal structure, which was first defined by Kashiwara using quantum groups. The main reference of this section is [BK, Section 5].

Let I be a finite set of indices. Let Λ be a lattice and $\Lambda^\vee = \Lambda^*$ be its dual lattice, and let $\{\alpha_i : i \in I\}$ be a subset of Λ and $\{\alpha_j^\vee : j \in I\}$ be a subset of $\Lambda^\vee$. Denote by $\mathfrak{g}$ the Kac-Moody algebra associated to the Cartan matrix $A = (a_{ij})_{i,j\in I}$ with $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$, where $\langle \cdot, \cdot \rangle$ is the evaluation pairing. Also denote by e_i, f_i $i \in I$ the Chevalley generators of $\mathfrak{g}$. We say a $\mathfrak{g}$-module V is *an integrable highest weight module* if:

- V admits a weight decomposition $V = \bigoplus V_\lambda$ and the weights are bounded above.
- e_i and f_i act locally nilpotently for $i \in I$, i.e., for any $v \in V$ and any $i \in I$, there exists an integer N such that $e_i^N(v) = 0$ and $f_i^N(v) = 0$.

For a non-zero vector $v \in V$ and $i \in I$, denote by $h_{i+}(v)$ the smallest positive integer j such that $e_i^{j+1}(v) = 0$ and we use the convention $h_{i+}(0) = -\infty$ for $v = 0$. Similarly $h_{i-}(v) = \min\{j \in \mathbb{Z} : f_i^{j+1}(v) = 0\}$. Further, denote $d_i(v) := h_{i+}(v) + h_{i-}(v) + 1$ to be the maximal dimension of the irreducible $\mathfrak{sl}_2$-submodule in $U(\mathfrak{g}_i)v$, where $\mathfrak{g}_i$ is the subalgebra of $\mathfrak{g}$ generated by e_i, f_i and $h_i = [e_i, f_i]$.

For each $i \in I$ and $d \geq 0$, define the subspace

$$V_i^{<d} := \{v \in V : d_i(v) < d\}.$$

We say that a basis $\mathbf{B}$ of a integrable highest weight $\mathfrak{g}$-module V is a *weight basis* if $\mathbf{B}$ is compatible with the weight decomposition, i.e., $\mathbf{B}_\lambda := V_\lambda \cap \mathbf{B}$ is a basis of V_λ for any λ being a weight of V.

Definition 2.1. *We say that a weight basis $\mathbf{B}$ in an integrable highest weight $\mathfrak{g}$-module V is* perfect *if for each $i \in I$ there exist maps $\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \cup \{0\}$ such that $\tilde{e}_i(b) \in \mathbf{B}$ if and only if $e_i(b) \neq 0$, and in the latter case on has*

$$e_i(b) \in \mathbb{C}^\times \cdot \tilde{e}_i(b) + V_i^{<d_i(b)}; \tag{3}$$

and $\tilde{f}_i(b) \in \mathbf{B}$ if and only if $f_i(b) \neq 0$, and in the latter case on has

$$f_i(b) \in \mathbb{C}^\times \cdot \tilde{f}_i(b) + V_i^{<d_i(b)}. \tag{4}$$

We refer to a pair $(V, \mathbf{B})$, where V is an integrable highest weight $\mathfrak{g}$-module and $\mathbf{B}$ is a perfect basis of V, as a based *$\mathfrak{g}$-module.*

Denote by V^+ the space of the *highest weight vectors* of V:

$$V^+ = \{v \in V : e_i(v) = 0,\ \forall i \in I\}.$$

Denote $\mathbf{B}^+ := \mathbf{B} \cap V^+$. Then we have the following result.

Proposition 2.2. *For any perfect basis $\mathbf{B}$ for V, the subset $\mathbf{B}^+$ is a basis for V^+.*

Proof. For $v \in V^+$, $e_i(v) = 0$, $\forall i \in I$. $\mathbf{B}$ is a basis of V, so $v = \sum_{b\in\mathbf{B}} \alpha_b b$ with $\alpha_b \in \mathbb{C}$. Therefore

$$e_i(v) = \sum_{b\in\mathbf{B}} \alpha_b e_i(b) = \sum_{b\in\mathbf{B}, e_i(b)\neq 0} \alpha_b e_i(b) = 0.$$

$\mathbf{B}$ is perfect so by equation (3), if $e_i(b) \neq 0$ then for some $x_b \in V_i^{<d_i(b)}$ and $\beta_b \in \mathbb{C}^\times$,

$$e_i(b) = \beta_b \tilde{e}_i(b) + x_b.$$

Hence

$$\sum_{b\in\mathbf{B}, e_i(b)\neq 0} (\alpha_b\beta_b\tilde{e}_i(b) + \alpha_b x_b) = 0.$$

Take $n = \max\{h_{i+}(\tilde{e}_i(b)) : b \in \mathbf{B}, e_i(b) \neq 0\}$ and $\mathbf{B}_n := \{b \in \mathbf{B} : \alpha_b \neq 0, h_{i+}(\tilde{e}_i(b)) = n\}$. Then

$$e_i^n(e_i(v)) = 0 = \sum_{b\in\mathbf{B}_n} \alpha_b\beta_b e_i^n(\tilde{e}_i(b)).$$

Note that for any $b \in \mathbf{B}_n$, $\beta_b \neq 0$ and $e_i^n(\tilde{e}_i(b)) \neq 0$. So $\alpha_b = 0$ and $\mathbf{B}_n$ is empty. So for any $b \in \mathbf{B}$ such that $\alpha_b \neq 0$, $h_{i+}(\tilde{e}_i(b)) = 0$. So $h_{i+}(b) = 0$ and $b \in \mathbf{B}^+$. $\square$

3. PERFECT BASIS IN $[\mathcal{C}]$

Recall from [Si, Section 2.5], if given $q \neq 1$ being a primitive lth-root of unity in $\mathbb{F}$ and $\mathbf{q} = (q_0, \cdots, q_{l-1}) \in \mathbb{F}^l$ with $q_i = q^{k_i}$ for $k_i \in \mathbb{Z}/l\mathbb{Z}$, we can construct an $\widehat{\mathfrak{sl}_l}$-categorification on $\mathcal{C} = \bigoplus_{n\geq 0} H_n - \text{mod}$, where $H_n = H_{\mathbb{F},q,\mathbf{q}}(n)$ denotes the cyclotomic Hecke algebra, which is the quotient of the affine Hecke algebra $\mathcal{H}_q^{\text{aff}}(n)$ by the extra relation $(X_1 - q_0)\cdots(X_1 - q_{l-1}) = 0$ (which is also called a cyclotomic polynomial). The categorification data is given as follows:

- The biadjoint endofunctors $E = \bigoplus \text{Res}_n^{n+1}$ and $F = \bigoplus \text{Ind}_n^{n+1}$, with the decompositions $E = \bigoplus_{i-0}^{l-1} E_i$ and $F = \bigoplus F_i$, where E_i is the i-Restriction and F_i is the i-Induction, defined in [Si, Section 2.4].
- $L = \bigoplus L_n \in \text{End}(E)$ with L_n denoting the n-th Jucys-Murphy element in H_n.
- $T = \bigoplus T_{n-1} \in \text{End}(E^2)$ with $T_{n-1} \in H_n$ being a particular generator of the cyclotomic Hecke algebra.

For $i = 0. \cdots, l-1$, $[E_i]$ and $[F_i]$ define a $\mathfrak{sl}_2$-action on $[\mathcal{C}] = K_0(\mathcal{C}) \otimes \mathbb{C}$. It is mentioned in [Si, Proposition 3.4] that we have the weight decomposition $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$, where $\mathcal{C}_\lambda$ is the full subcategory of $\mathcal{C}$ consisting of objects whose class is in the weight space $[\mathcal{C}]_\lambda$.

The reason why we are interested in crystals is that the categorical $\widehat{\mathfrak{sl}_l}$ action on $\mathcal{C}$ gives rise to a canonical crystal structure on the set $\text{Irr}\mathcal{C}$ of simple objects in $\mathcal{C}$. In this section, we are going to construct a perfect basis for the $\widehat{\mathfrak{sl}_l}$-module $[\mathcal{C}]$ using results in Propostion 1.1, and deduce that $[\mathcal{C}]$ is an irreducible $\widehat{\mathfrak{sl}_l}$-module.

Denote $V = [\mathcal{C}]$. According to the weight decomposition, V is an integrable highest weight $\mathfrak{g}$-module. Take the basis $\mathbf{B}$ of $V = [\mathcal{C}]$ consisting of classes of all simple objects. Similarly to Equation (1) and (2), we can define maps $\tilde{e}_i, \tilde{f}_i : \text{Irr}\mathcal{C} \to \text{Irr}\mathcal{C} \sqcup \{0\}$ for $i \in I$. Note that for a simple object S in $\mathcal{C}$, $\tilde{e}_i(S) = 0$ if and only if $\operatorname{soc} E_i(S) = 0$, iff and only if $E_i S = 0$, i.e., $e_i[S] = 0$. Together with Proposition 1.1, we see that $\tilde{e}_i, \tilde{f}_i$ are maps satisfying conditions (3) and (4), so $\mathbf{B} = \text{Irr}\mathcal{C}$ is a perfect basis of V and $(V, \mathbf{B})$ is a based $\mathfrak{g}$-module.

Now consider the basis $\mathbf{B}^+$ of the space of highest weight vectors. $[S] \in \mathbf{B}^+$ means that S is simple and $\tilde{e}_i([S]) = 0$ for all $i \in I$. Then $e_i[S] = 0$, which means exactly $E_i S = 0$ for all $i \in I$. So $ES = \bigoplus E_i S = 0$, i.e., $\bigoplus \text{Res}_{n-1}^n S = 0$ for all $n \geq 0$. The only simple S in $\mathcal{C}$ is a simple H_0-module. Since $H_0 = \mathbb{F}$, so $S \simeq \mathbb{F}$ is unique up to isomorphism. $[S]$ is the unique (up to scalar) highest weight vector in V. Therefore V is irreducible.

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