Categorical diagonalization

(B. Elias - M. Hogencamp)

F operator, diagonalizable if there are numbers $\lambda_i \neq \lambda_j$ such that $\prod_{i=1}^{n} (F - \lambda_i) = 0$

$$p_i = \prod_{j \neq i} \frac{(F - \lambda_j)}{(\lambda_i - \lambda_j)}$$

<u>Fact</u>: 1) $p_i^2 = p_i$ orthogonal idempotents

$p_i p_j = 0 \quad i \neq j$

2) $\sum p_i = 1 \qquad F p_i = \lambda_i p_i$

p_i are projectors onto eigenspaces.

Elias-Hogencamp categorify these.

$\mathcal{C}$ = graded tensor category

$F \in K^-(\mathcal{C})$

htpy category of complexes in $\mathcal{C}$

eigenvector, eigenvalue, diagonalizable = ?

Def: P is an eigenobject for F if there is a morphism

$$\alpha : \mathbb{1}[\lambda] \to F$$

such that

grading shift

$$\alpha \otimes \mathrm{Id}_P : P[\lambda] \xrightarrow{\simeq} F \otimes P$$

is an isomorphism (in htpy category),

λ eigenvalue
α eigenmap

$$(F - \lambda) \cdot P = 0$$

$$\mathrm{Cone}(\alpha) \otimes \mathrm{Id}_P \simeq 0$$

Def: F is diagonalizable if there are maps $\alpha_0, \dots, \alpha_n$

$$\alpha_i : \mathbb{1}[\lambda_i] \longrightarrow F$$

such that

$$\bigotimes \mathrm{Cone}(\alpha_i) \simeq 0$$

categorifies $\prod (F - \lambda_i) = 0$

Thm (EH): 1) If F is diagonalizable all λ_i have different homol. grading then there exist $P_i \in K^-(\mathcal{C})$ s.t. $\alpha_i \otimes Id_{P_i} : P_i[\lambda_i] \xrightarrow{\sim} F \otimes P_i$ iso
($\Rightarrow P_i$ are eigenobjects)

2) $P_i^2 \simeq P_i$, $P_i P_j \simeq 0 \quad i \neq j$

3) $\mathbb{1} = (P_1 \to P_2 \to \dots \to P_n)$
ordered by homol. deg. λ_i

($\Rightarrow$ can get semiorthogonal decomposition of our category)

Sketch of the proof:

1) $\bigotimes_{j \neq i} \mathrm{Cone}(\alpha_j) = Q_i$ eigenobject for F

bounded

Q_i = categorification of numerator $\prod_{j \neq i} (F - \lambda_j)$ of P_i

2) To construct P_i, assume $n=1$
α_0, α_1

There are two eigenobjects

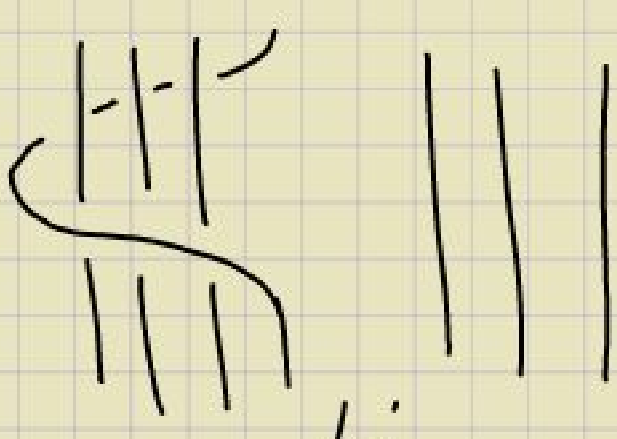
$$
P_0 = \begin{tikzcd}
\mathbb{1} \arrow[r, "\alpha_0"] \arrow[dr, "\alpha_1"] & F \\
\mathbb{1} \arrow[r, "\alpha_0"] \arrow[dr, "\alpha_1"] & F \\
\mathbb{1} \arrow[r] \arrow[dr, "\alpha_1"] & F \\
\vdots & \vdots
\end{tikzcd}
$$

categorifies $\dfrac{F - \lambda_0}{\lambda_1 - \lambda_0}$ where we expand denominator into infinite geometric series

$P_1 = \mathrm{Cone}(\mathbb{1} \xrightarrow{\alpha_1} P_0)$

We care about this because we want to diagonalize something in some category

Br_n



L_i

$L_i L_j = L_j L_i$

$\Rightarrow L_i$ generate a commutative subalgebra in H_n

($\Rightarrow$ we can simultaneously diagonalize them)

$$\mathbb{C}[S_n] = \bigoplus_{|\lambda|=n} m_\lambda V_\lambda$$

$m_\lambda = \dim V_\lambda$
$= \#SYT$ of shape λ

$$\boxed{\begin{array}{l} P_1 \oplus P_2 \oplus P_3 \\[1em] d = \begin{pmatrix} 0 & 0 & 0 \\ d_{12} & 0 & 0 \\ d_{13} & d_{23} & 0 \end{pmatrix} \qquad d^2 = 0 \end{array}}$$

This picture has a q-deformation

$$H_n = \bigoplus_{|\lambda|=n} m_\lambda V_\lambda$$

irrep of H_n

<u>Fact</u>: L_i diagonalize simultaneously and the basis of projectors $\{p_T\}$ where T runs over all SYT.

$L_i \, p_T = q^{c_i(T)} \, p_T$ $\quad \begin{array}{|c|c} \hline 5 & \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$ cont. $\begin{array}{|c|c|} \hline -2 & \\ \hline -1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$

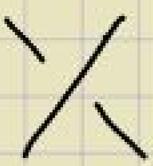
$c_i(T)$ = content of a box labelled by i in T

eigenvalue of F_k $= \Sigma$ contents in $T(i)$

<u>Elias – Hogencamp categorify this story with one subtlety</u>

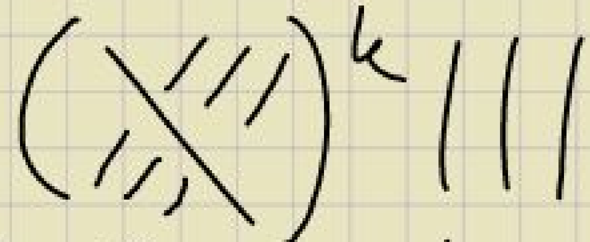
<u>Recall</u>: H_n is categorified by the category of Soergel bimodules $SBim_n$

elem braid $\to$ complex of Soergel bimodules



$\beta \rightsquigarrow$ product of those

$L_i \rightsquigarrow$ very precise complexes

$F_k = L_1 \cdots L_k =$ 

full twist on the first k strands

L_i commute $\Rightarrow$ F_k commute as well

Not enough maps to categorify L_i, but enough maps to categorify F_k!

<u>Thm (EH)</u>: F_k diagonalize simultaneously in the categorical sense

$$\text{Eigenvalues} = q^{\sum x_i} \, t^{\sum y_i}$$

$(x_i, y_i) =$ boxes in T with labels $\leq k$

Here (q, t) are related to (internal grading, homological grading) by some monomial change

Another example of diagonalization

X = alg variety
F = line bundle on X
$\mathcal{C} = D^b\mathrm{Coh}(X)$

$\alpha_i : \mathcal{O}_X \to F$ sections of F
$\mathcal{O}_X = \mathbb{1}$

$\bigotimes_i \mathrm{Cone}(\alpha_i) \simeq 0$ all α_i vanish

$\mathrm{Cone}(\alpha_i) = \mathrm{Cone}(\mathcal{O}_X \xrightarrow{\alpha_i} F) = \mathcal{O}_{\{\alpha_i = 0\}}$

<u>Q</u>: When is F diagonalizable in a categorical sense?

<u>Lemma:</u> $\bigotimes \mathrm{Cone}(\alpha_i) \simeq 0$

$$\Updownarrow$$

$$\forall x \in X \quad \exists i : \quad \alpha_i(x) \neq 0$$

$$\Updownarrow$$

α_i do not vanish simultaneously

$$\Updownarrow$$

$$X \xrightarrow{i} \mathbb{P}^n$$
$$x \longrightarrow [\alpha_0(x) : \ldots : \alpha_n(x)]$$

$i^* \mathcal{O}(1) = F$ ($\Leftrightarrow F$ is generated by sections)

Question: What are the eigenobjects in these situations?

Torus action w/ fin many fixed points
$\rightsquigarrow$ grading on category

Graded version

$X \circlearrowleft \mathbb{C}^*$

$D^b\mathrm{Coh}(X)$ is graded by characters of $\mathbb{C}^*$

$\alpha_i : \mathcal{O}_X[\lambda_i] \to F$

$\mathbb{C}^*$-equivariant sections

Lemma: F is diagonalizable with eigenvalues λ_i

$\Updownarrow$

$\mathbb{C}^* \circlearrowright X \xrightarrow{i} \mathbb{P}^n \circlearrowleft \mathbb{C}^*$

$\mathbb{C}^*$ acts with weights $\lambda_0, \dots, \lambda_n$ ($\rightsquigarrow$ eigenvalues)

s.t. $i^*\mathcal{O}(1) = F$, i is $\mathbb{C}^*$-equivariant (sections = pullbacks of standard coord sections of $\mathcal{O}(1)$)

Higher Ext's: slightly different diagonalization

$$\alpha_i \in \mathrm{Ext}^i(\mathbb{1}, F)$$

map $F \to \mathbb{1}$ categorifies λ_i in a different way

Localization formula:

trace $= \sum$ eigenvalues

Thm (Neguţ, Rasmussen)

$\mathcal{C}$ graded tensor category

$F \in K^-(\mathcal{C})$

F is diagonalizable

$\Updownarrow$

there is a pair of adjoint functors

$$K^-(\mathcal{C}) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^b \mathrm{Coh}_{\mathbb{C}^*}(\mathbb{P}^n)$$

s.t. $i^* \mathcal{O}(1) = F$

Instructive for main conjecture

Geom meaning of eigenobjects,
strategy of proof

This is published in recent paper
1608.07308

Произвольно умножить точку на
точку ⇝ внешняя алгебра

Idea of proof:

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathrm{Hom}(1, F^{\otimes k})$$

graded algebra with an action of

$$\mathbb{C}[\alpha_0, \alpha_1, \dots, \alpha_n]$$

polynomial algebra

$$\alpha_i \in \mathrm{Hom}(\mathbb{1}, F)$$

$M \in K^-(\mathcal{E})$

$$\bigoplus_{k=0}^{\infty} \mathrm{Hom}(M \otimes F^k)$$

graded $\mathcal{A}$-module

graded $\mathbb{C}[\alpha_0, \alpha_1, \dots, \alpha_n]$-module

$\rightsquigarrow$ sheaf on $\mathbb{P}^n$

$i_* M$

Koszul complex for $\mathbb{P}^n$

with coordinates $[z_0 : \dots : z_n]$

Koszul complex for $z_0, \dots, z_n$

$$= \bigotimes \mathrm{Cone}(z_i)$$

$$= \bigotimes [\mathcal{O} \xrightarrow{z_i} \mathcal{O}(1)] \simeq 0 \quad \text{in } D^b\mathrm{Coh}(\mathbb{P}^n)$$

$$z_i : \mathcal{O} \to \mathcal{O}(1)$$

But $\otimes \operatorname{Cone}(\alpha_i) = 0$

$\Rightarrow i_* M$ is really defined on $\mathbb{P}^n$ and not on $\mathbb{C}^{n+1}/\mathbb{C}^*$

To construct i^*:

Beilinson:

$D^b \operatorname{Coh}(\mathbb{P}^n) \simeq$ homotopy category of complexes constructed out of $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$

$\wr\wr$

homotopy category of complexes constructed of all $\mathcal{O}(k)$ modulo relation $\otimes \operatorname{Cone}(z_i) = 0$

We can write Koszul complex as

$$\mathcal{O} \to (n+1)\mathcal{O}(1) \to \binom{n+1}{2}\mathcal{O}(2) \to \dots \to \mathcal{O}(n+1)$$

$i^* \mathcal{O}(k) = F^k \quad \forall k$

Arbitrary complex on $\mathbb{P}^n$: resolve by $\mathcal{O}(k)$

(grading on image of i^* comes from homol grading of $D^b Coh_{\mathbb{C}^*}(\mathbb{P}^n)$)

Pushforward via map to $\mathbb{P}^n$ is constructed abstractly.

What are the eigenobjects (eigenprojectors) in this construction?

Eigenprojectors for F $P_i = i^* \left(\begin{matrix} \text{eigenprojectors} \\ \text{for } \mathcal{O}(1) \text{ on } \mathbb{P}^n \end{matrix} \right)$

In K-group of $\mathbb{P}^n$, $K_{\mathbb{C}^*}(\mathbb{P}^n)$,
eigenvectors of $\mathcal{O}(1)$
= torus-fixed points

weight = weight of fiber of line bundle at point

By Localization Theorem,
$$K_{\mathbb{C}^*}(\mathbb{P}^n) = \bigoplus K(\text{fixed points})$$

$K_{\mathbb{C}^*}(\mathbb{P}^n)$

eigenprojectors = multiples of fixed points

$$\mathcal{O}_p \overset{L}{\otimes} \mathcal{O}_p = \mathcal{O}_p \otimes \Lambda^{\cdot} T_p^*$$

look at proof of theorem about diagonalization and construction of P_i:

$\mathbb{P}^1$, $\mathcal{O}(1)$

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "z_0"] \arrow[dr, "z_1"] & \mathcal{O}(1) \\
\mathcal{O} \arrow[r, "z_0"'] \arrow[dr, "z_1"'] & \mathcal{O}(1) \\
\arrow[r] \arrow[dr] & \mathcal{O}(1) \\
& \cdots
\end{tikzcd}$$

$$\cong \mathcal{O} \otimes \mathbb{C}[t] \xrightarrow{z_0 + t z_1} \mathcal{O}(1) \otimes \mathbb{C}[t]$$

$\mathbb{P}^1 \times \mathbb{A}^1_t$, $\pi_*\left[\mathcal{O} \xrightarrow[z_0 + t z_1]{} \mathcal{O}(1)\right]$

$$= \pi_* \mathcal{O}_{\{z_0 + t z_1 = 0\}}$$

$$= j_* \mathcal{O}_{\{z_1 \neq 0\}}$$

$$\mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{\pi} \mathbb{P}^1$$

get multiple of str sheaf of pt in K-grp (eigenobject)

Important philosophy:

Eigenvectors $\leftrightarrow$ fixed points

Approaching the big conjecture:

Main Conjecture:

$$K^-(SBim_n) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^b Coh(FH_n)$$

flag Hilbert scheme

$i^* \mathcal{L}_i = L_i$ commuting braids

line bundles on FH_n

FH_n much more complicated than $\mathbb{P}^n$

$$FH_n = \{ \mathbb{C}[x,y] \supset I_1 \supset \dots \supset I_n, \ I_k = \text{ideals in } \mathbb{C}[x,y] \text{ supported on } \{y=0\} \}$$

Have map

$$FH_n \to FH_{n-1} \times \mathbb{C}$$

$$FH_n \xrightarrow{p} FH_{n-1} \times \mathbb{C}$$

$$(I_1 \supset \dots \supset I_n) \mapsto (I_1 \supset \dots \supset I_{n-1}) \times \left({}^{I_{n-1}}/_{I_n} \right)$$

pt

<u>Fact</u>: The fibers of p are projective spaces [of different dimensions]
(linear inside a projective bundle)

More precisely,

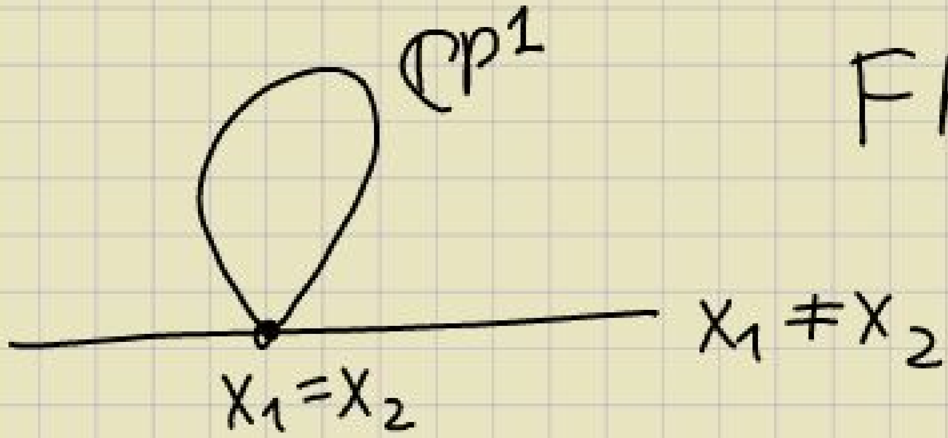
$$FH_n = \mathrm{Proj}_{FH_{n-1} \times \mathbb{C}} \left(\mathcal{E}_n^\vee \right)$$

where $\mathcal{E}_n$ = some explicit complex

Dim of fiber $\approx$ # generators of I_{n-1}

$$FH_2 \longrightarrow \underset{\underset{\mathbb{C}}{\shortparallel}}{FH_1} \times \mathbb{C}$$

$$(I_1 \supset I_2) \longrightarrow \frac{\mathbb{C}[x,y]}{I_1}, \frac{I_1}{I_2}$$



FH_2

Fact:

$$\mathcal{E}_n = \left[\begin{tikzcd} & T_n \arrow[dr, "-y"] & \\ & \oplus & \\ T_n \arrow[uur, "x-x_n"] \arrow[r, "y"'] & T_n \arrow[r, "x-x_n"] & T_{n-1} \\ & \oplus & \\ & \mathcal{O} \arrow[uur, "1"'] & \end{tikzcd} \right]$$

4-term complex

$$T_{n-1} = \frac{\mathbb{C}[x,y]}{I_{n-1}}$$ tautological vector bundle

dg-algebra
coord ring = symmetric algebra of $\mathcal{E}_n$

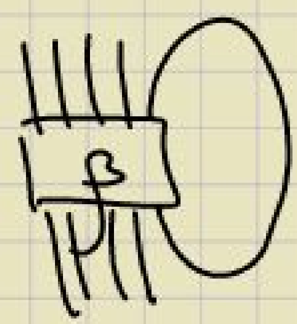
($\mathcal{E}_n$ quasi isomorphic to a complex of vector bundles)

Strategy of the proof:

Have a tower

$$\begin{tikzcd}
FH_n = \mathrm{proj}(\mathcal{E}_n) \arrow[d] & & K^-(SBim_n) \arrow[d, shift left, "Tr"] \arrow[dll, shift right] \\
FH_{n-1} \times \mathbb{C} \arrow[urr, shift right] \arrow[rr, shift left, "i^*"] & & K^-(SBim_{n-1}) \arrow[u, shift left, "I"] \arrow[ll, shift left, "i_*"]
\end{tikzcd}$$

$I(\beta) =$ 

$Tr(\beta) =$ 

Use relative version of Thm
to lift i_*, i^* to FH_n

(relies on properties of $\mathcal{E}_n$
which are hard:
need computations on Soergel side)

Consequences of Main Conjecture:

$$KhR(\beta) = \mathrm{Hom}_{SBim}(\mathbb{1}, \beta) =$$

$$= \mathrm{Hom}_{FH_n}(\mathcal{O}, i_* \beta)$$

$$= H^*(FH_n, i_* \beta)$$

<u>Ex:</u> $i_*(L_1^{a_1} \cdots L_n^{a_n}) = \mathcal{L}_1^{a_1} \cdots \mathcal{L}_n^{a_n}$

$\exists$ algorithm by induction to compute this explicitly for all a_i's

In particular for all braids that are closures of $L_1^{a_1} \cdots L_n^{a_n}$ this can be done

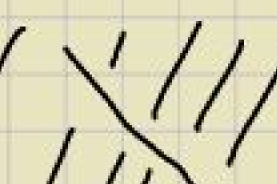
In particular

<u>Elias–Hogencamp</u>: Explicit computation of $L_1 \cdots L_n$ agrees with Conj numerically up to $n=6$

They have this for general braids $\rightsquigarrow$ hope to use this in general

$T(m,n) \rightsquigarrow$  $\cdot\, L_1^{a_1} \cdots L_n^{a_n}$

$$a_i = \left[\frac{im}{n}\right] - \left[\frac{(i-1)m}{n}\right]$$

$i_*\big($  $\big) = \mathcal{O}_{FH_n(\mathbb{C}^2,0)}$

Restrict line bundles to punctual flag Hilbert scheme

Thm (G, Negut) This agrees with "refined Chern-Simons invariants" of Aganagic-Shakirov, Cherednik.

(their invts agree w/torus knots)

This sheaf carries action of $\mathbb{C}^* \times \mathbb{C}^*$

cohomology of $T(m,n)$ carries action of Cherednik algebra

$$FH_n \to H_n$$

quantize $\rightsquigarrow L_{\frac{m}{n}}$

Related to rational Cherednik algebra

(Bezrukavnikov knows how to prove this?)

Before no one expected nice answers for homology of torus knots.

Other consequences:

Eigenobjects for L_i in SBim

= categorical Jones-Wenzl projectors P_T

in FH_n

they correspond to fixed points T of torus action

also labelled by standard tableaux

Consequence of main conjecture:

$$\underbrace{\mathrm{End}(P_T)}_{\text{algebra}} \simeq \underbrace{\mathrm{Hom}_{SBim}(\mathbb{1}, P_T)}_{\text{v. space}} \simeq \underbrace{\text{local}}_{\text{coordinate algebra of } FH_n \text{ at } T}$$

(follows from identification of fixed points as projectors)

Thm (Hogencamp)

$$\mathrm{End}(P_{(n)}) = \mathbb{C}[u_1, \dots, u_n]$$

local (smooth) coord ring of FH_n at (n)

($\otimes$ exterior algebra to account for a-grading: need more things on Hilbert scheme)

Thm (Abel, Hogencamp)

$$\mathrm{End}_{a=0}(P_{(1\dots1)}) = \frac{\mathbb{C}[x_1, \dots, x_n, y_{ij}]}{[X,Y]=0}$$

comm alg, not free

local coord ring of FH_n at $(1\dots1)$

(ADHM descr. of Hilb + slice to action at a point)

$$X = \begin{pmatrix} x_1 & & 0 \\ 1 & \ddots & \\ 0 & 1 & x_n \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ y_{ij} & & 0 \end{pmatrix}$$

$$n=2 \quad \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$$

relation: $y(x_1 - x_2) = 0$

$\rightsquigarrow$ HH^* of projectors
= rings built of commuting matrices

$\rightsquigarrow$ homology of products of L_i's
sheaf for figure 8

Symmetry in KhR^*
from swapping x and y

Sheaves sitting on 0 point are equivariant

Knot supported on punctual Hilbert scheme

$FH_n(\mathbb{C}^2, 0)$ symmetric in x, y
using facts about the Soergel category

Symmetry of eigenmaps
(q,t swapped)
For links, more subtle

From general module construction, symmetry $X \leftrightarrow Y$ not clear

Even for single crossing symmetry is not apparent upstairs

Тавтологическое расслоение
в категории Зёргеле

Прорези $\rightsquigarrow$ бимодуль Зёргеле

Производная категория схем Гильберта

Внешние степени

Пучки на флаг схеме гильберта

центр

симм полиномы

функторы Шура от тавтологического расслоения

Центр категории Зёргеля
BFO

не связан с Hilb

ф-р Шура

центр = производ кат Hilb