Gorsky - 1

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Imprecise conjecture: Given a braid β on n strands, one can construct a sheaf (or a compl. of sheaves) $\mathcal{F}_\beta$ on some algebraic variety FH_n (flag Hilbert scheme)

(Such that Khovanov-Rozansky homology of $\overline{\beta}$ = triply-graded vector space, topological invt of β, generalizes HOMFLY polynomial.)

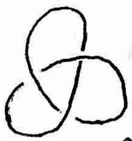
$$\|$$

sheaf cohomology of $\mathcal{F}_\beta$. ~~for~~ rather

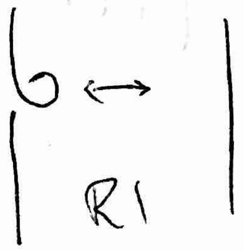
Application: can compute RHS explicitely using algebraic geometry.

0. Reminder about knots & braids

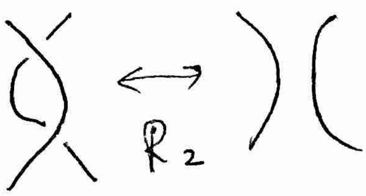
Th. (Reidemeister)



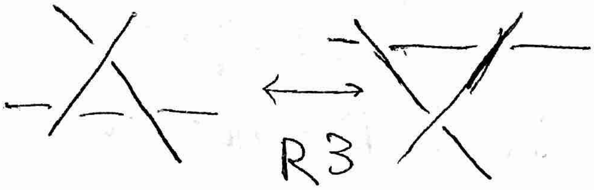
Two diagrams represent the same knot (or link) if they're connected by a sequence of moves



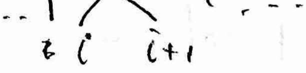
R1

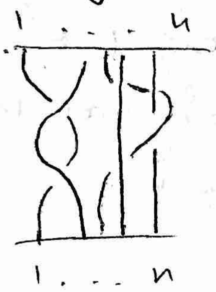


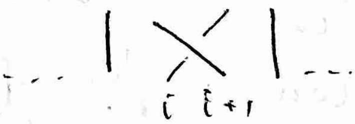
R_2



Cor. Top. invt of a knot = a combinatorial invariant of a diagram, which does not change in R1–3.

Braid group: Generators  $= \sigma_i$



 $= \sigma_i^{-1}$.

generators.

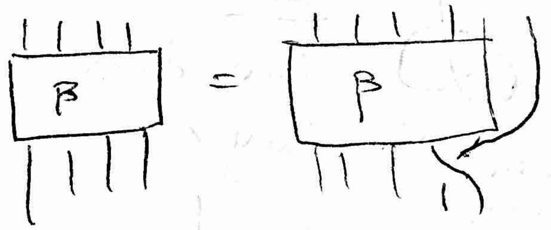
$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1 \qquad (R2)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (R3)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2.$$

Theorem (Alexander): every knot is a closure of some braid.

Theorem (Markov): two braids represent the same knot, if they are related by a sequence of moves $\beta \longleftrightarrow \alpha \beta \alpha^{-1}$,



① HOMFLY polynomial & Hecke algebra

- H_n = algebra w. generators $T_1, \dots, T_{n-1}$ & relations
$(T_i - 1)(T_i + q) = 0$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,
$T_i T_j = T_j T_i$, $|i-j| \geq 2$.

Facts: 1) There's a homomorphism $\mathbb{C}[Br_n] \to H_n$

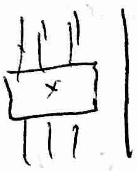
2) at $q = 1$, we get $\mathbb{C}[S_n]$.

Assume: q is a formal parameter.

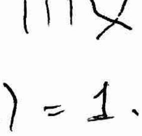
3) $\dim H_n = n!$, there's a basis T_w
in H_n: $w \in S_n$, $w = s_{i_1} \dots s_{i_n}$ – reduced expression.
$T_w = T_{i_1} \dots T_{i_n}$.

- Thm (Ocneanu, Jones) There's a unique linear functional $Tr_n : H_n \to \mathbb{C}(q, z)$ such that the following holds.

$H_n \xhookrightarrow{i} H_{n+1}$

(a) $Tr(i(x)) = Tr(x)$  $= i(x)$

(b) $Tr(ab) = Tr(ba)$

(c) $Tr($  $) = z\,Tr(x)$

(d) $Tr(1) = 1$.

Idea of proof:

Every element in H_{n+1} can be presented as $x = y_1 + y_2 T_n y_3$, where $y_1, y_2, y_3 \in H_n$.

In reduced expressions for w, we can assume that T_n appears at most once.

$$Tr(x) = Tr(y_1) + Tr(y_2 T_n y_3) = Tr(y_1) + Tr(y_3 y_2 T_n) =$$
$$= Tr(y_1) + Tr(y_3 y_2).$$

Different pf #1:

V = rep of H_n. $Tr(x) := Tr(X|_V)$

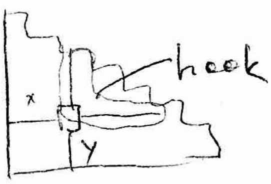
Fact.: Irred. reps of H_n are labeled by Young diagrams: V_λ, $|\lambda| = n$.

$Tr(x) = \sum a_\lambda Tr(x|_{V_\lambda})$ – satisfies (b) automatically.

There's unique choice of a_λ s.t. (a-d) are satisfied:

$$a_\lambda = \prod_{\square \in \lambda} \frac{q^x w - q^y z}{1 - q^{h(\square)}}, \quad h(\square) = \text{hook length}$$

$$w = 1 - q + z.$$



<u>Remark</u>: $a = \frac{w}{z}$ is more natural.

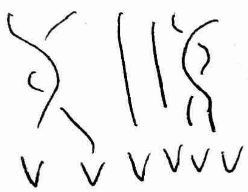
$$z = -\frac{1-q}{1-a}.$$

<u>Different proof #2:</u>

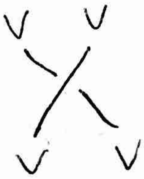
Reshetikhin–Turaev construction

$V = \mathbb{C}^N \leftarrow$ representation of a quantum group $\mathcal{U}_q\mathfrak{gl}_N$.

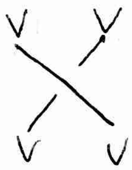
$$R : V \otimes V \to V \otimes V$$



– braid



$$\begin{array}{c} V\otimes V \\ \uparrow R \\ V \otimes V \end{array}$$



$$\begin{array}{c} V\otimes V \\ \uparrow R^{-1} \\ V \otimes V \end{array}$$

Satisfies braid relations.

$$V^{\otimes n} \xrightarrow{R_\beta} V^{\otimes n}, \quad \beta \text{ - braid.}$$

$$\mathrm{Tr}(R_\beta) = P_N(q)$$

<u>Fact</u>: $P_N(q) \underset{\text{related to}}{\approx} \mathrm{Tr}(q, a = q^N)$.

Fact: (quantum Schur-Weyl duality).

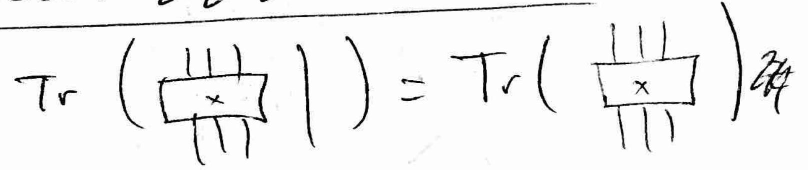
$$V^{\otimes n} = (\mathbb{C}^N)^{\otimes n} = \bigoplus_{|\lambda|=n} \underbrace{V_\lambda}_{\substack{\text{irrep} \\ \text{of } H_n.}} \otimes \underbrace{\mathcal{U}_\lambda}_{\text{rep. of } U_q gl_N}$$

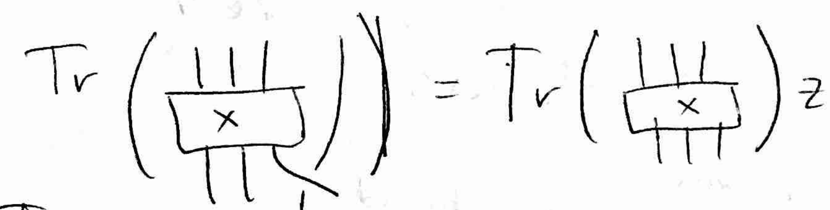
R_β acts in V_λ and preserves $\mathcal{U}_\lambda$

$$Tr(R_\beta) = \sum_\lambda Tr(\beta|_{V_\lambda}) \cdot \dim_q \mathcal{U}_\lambda$$

$\dim_q \mathcal{U}_\lambda$ = same as a_λ above, given by the product formula.

Recall ~~Markov thm~~:

(a) 

(c) 

(d) $Tr(1) = Tr(1\,|)$

(b) $Tr(ab) = Tr(ba)$.

$$P(\beta) = \underset{\text{depends on \# of strands}}{c_1}\, c_2^{\,w(\beta)}\, Tr(\beta)$$

$$w(\beta) : Br_n \to \mathbb{Z}$$
$$\sigma_i \mapsto 1$$
$$\sigma_i^{-1} \mapsto -1.$$

$$c_1 = \left(-\frac{1-a}{(1-q)\sqrt{a}}\right)^{n-1}$$

$$c_2 = \sqrt{a}$$

HOMFLY homology

$(T_i - 1)(T_i + q) = 0 \Leftrightarrow T_i$ has eigenvalues $1, -q$.

$b_i = T_i + q$, has eigenvalues $1+q, 0$.

$b_i^2 = (1+q) b_i$.

Exercise: Braid relations for $T_i \Leftrightarrow b_i b_{i+1} b_i - q b_i =$

$= b_{i+1} b_i b_{i+1} - q b_{i+1}$.

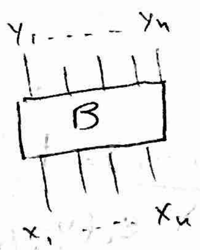
There's an element $b_{i,i+1}$ such that

$\boxed{b_i b_{i+1} b_i = q b_i + b_{i,i+1}}$ and $\boxed{b_{i+1} b_i b_{i+1} = q b_{i+1} + b_{i,i+1}}$

Rmrk: $b_i, b_{i,i+1}$ are related to Kazhdan-Lusztig basis.

$R = \mathbb{C}[x_1, \dots, x_n]$

We will consider $R-R$ bimodules.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$
modules over $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$.



There is a natural tensor product on bimodules: $B_1 \otimes B_2 = B_1 \underset{\mathbb{C}[y]}{\otimes} B_2$

$B_1 = \mathbb{C}[x, y]$-module, $B_2 = \mathbb{C}[y, z]$-module.

$$B_i = \frac{\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]}{\left(\begin{array}{l} x_i + x_{i+1} = y_i + y_{i+1} \\ x_i x_{i+1} = y_i y_{i+1} \\ x_j = y_j \text{ for } j \neq i,i+1 \end{array}\right)} = R \otimes_{R^{s_i}} R$$

s_i permutes x_i and x_{i+1}.

Lemma:

$$B_i \otimes B_i \simeq B_i \oplus B_i[1]$$ graded shift

(categorification of $b_i^2 = (1+q)b_i$).

Rmk. Common notation

$B_i =$ 

Idea of proof: WLOG, can assume

$n=2$; $x_1+x_2 = y_1+y_2 = 0$:

$$B = \frac{\mathbb{C}[x,y]}{(x^2=y^2)}$$

$$B \otimes B = \frac{\mathbb{C}[x,y,z]}{x^2=y^2=z^2} \quad \text{as a } \mathbb{C}[x,z] \text{ module}$$

$$\cong \frac{\mathbb{C}[x,z]}{x^2=z^2} \oplus y\frac{\mathbb{C}[x,z]}{x^2=z^2}.$$

Lemma: $$B_{i,i+1} = R \otimes_{R^{S_3}} R = \frac{\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]}{\left(\begin{array}{l} e_i(x_i,x_{i+1},x_{i+2}) = e_i(y_i,y_{i+1},y_{i+2}) \\ x_j = y_j,\ j \neq i,i+1 \end{array}\right)}$$

S_3 permutes x_i, x_{i+1}, x_{i+2}

Then $B_i \otimes B_{i+1} \otimes B_i = B_{i,i+1} \oplus B_i[1]$

As a result, B_i categorify b_i.

- ~~Def~~. Bott-Samelson bimodule is an arbitrary product $B_{i_1} \otimes B_{i_2} \otimes \dots \otimes B_{i_k}$.

Indec. Soergel bimodule is a direct summand of a BS bimodule.

Ex. B_i, $B_{i,i+1}, \dots$

Category of Soergel bimodules: additive category generated by direct sums of indec. Soergel bimodules with grading shifts. ($SBim_n$)

- Facts: 1) tensor product of Soergel bimodules is a Soergel bimodule.

2) (Soergel) There are exactly $n!$ indecomposable Soergel bimodules. (up to a shift).

Ex. $n=2$: R = identity bimodule, $\mathbb{C}[x_1, x_2, y_1, y_2] / x_i = y_i$, B_1

$n=3$: R, B_1, B_2, $B_1 \otimes B_2$, $B_2 \otimes B_1$, $B_{1,2}$.

3) (Soergel) K-split graded Grothendieck group ($SBim_n$) is isomorphic to H_n.

K-split graded Groth. group:

- Gens: [isom. classes of S. bim]

Relations: $[A \oplus B] = [A] + [B]$.

$[A[1]] = q[A]$.

(4) Indecomposable S. bimodules categorify the KL basis in H_n.

What about T_i?

$$T_i = b_i - q; \quad T_i^{-1} = q^{-1}(b_i - 1) \quad \text{in } H_n.$$

Rouquier complexes:

idea: interpret "−" as a homological shift.

$$[R[1] \to B_i] \quad T_i$$

$$[B_i \longrightarrow R] \quad T_i^{-1} \qquad \frac{\mathbb{C}[x,y]}{x^2=y^2} \to \frac{\mathbb{C}[x,y]}{x=y}$$

$$\varphi = (x_i - x_{i+1}) + (y_i - y_{i+1})$$

Exercise: φ is a correct morphism of bimodules

Thm (Rouquier)

T_i, T_i^{-1} satisfy braid relations up to a homotopy

$$T_i T_i^{-1} \simeq T_i^{-1} T_i \simeq R$$

$$T_i T_{i+1} T_i \simeq T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \ |i-j| \geq 2.$$

$$T_i T_i^{-1} = [B_i \to R] \otimes [R[1] \to B_i][-1]$$

$$= \left[B_i[-1] \to \begin{matrix} B_i^2 \\ \oplus \\ R[1] \end{matrix} \to B_i \right][-1] \simeq R.$$

using the relation $B_i^2 = B_i \oplus B_i[1]$.

For every braid β one can construct a complex of Soergel bimodules, which is well-defined up to a homotopy, which categorifies the projection of β to H_n.

<u>Theorem</u> (Khovanov) $\to M_i \to M_{i+1} \to \dots$ complex of ~~SB~~ Soergel bimodules.

$$H^\bullet\left[\dots \to R\mathrm{Hom}(R, M_i) \to R\mathrm{Hom}(R, M_{i+1}) \to \dots\right]$$

is a knot invariant, categorifies Tr.

<u>Rmk</u>: this agrees with an earlier construction of Khovanov-Rozansky, ~~which agrees~~ through matrix factorizations.

$$M \otimes R = M \otimes \frac{\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]}{x_i = y_i}$$

$M \otimes_L R$ = ~~same~~ same, but replace R by a free R-R-resolution.

Resolution of R b R-R-bimodules:

$$\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \xleftarrow{x_i - y_i} \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \oplus \dots \oplus$$

$$\mathbb{C}[x,y] \leftarrow \mathbb{C}[x_1,\dots x_n,y_1,\dots,y_n] \leftarrow \Lambda^2\mathbb{C}^n \otimes \mathbb{C}[x,y].$$

$$= \Lambda^1\mathbb{C}^n \otimes \mathbb{C}[x,y]$$

Koszul complex for the sequence $(x_i - y_i)$.

Gradings:

RHom grading $\longleftrightarrow$ a-grading.

Internal grading in M_i $\longleftrightarrow$ q-grading.

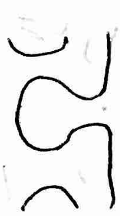
homological grading in M_i $\longleftrightarrow$ t-grading.

Another picture:

Jones / Temperley-Lieb / sl_2-Khovanov

$B_i \sim$ 

$T_i = [$  $\longrightarrow)($ $]$

$B_i B_{i+1} B_i =$  $= B_i$

$B_{i,i+1} = \emptyset$!