SOERGEL BIMODULES, HECKE ALGEBRAS, AND KAZHDAN-LUSZTIG BASIS

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Abstract. These are the notes for a talk at the MIT-Northeastern seminar for graduate students on category $\mathcal{O}$ and Soergel bimodules, Fall 2017.

Contents

1. Introduction

The main goal of this talk is to explain Soergel's approach to Kazhdan-Lusztig's conjecture [KL79]. This conjecture expresses the multiplicities of simple objects in standard ones in the principal block $\mathcal{O}_0$ of category $\mathcal{O}$ in terms of the values of certain polynomials in $\mathbb{Z}[v^{\pm 1}]$ at $v = 1$. These polynomials arise from Hecke algebras - certain algebras $\mathcal{H}$ over $\mathbb{Z}[v^{\pm 1}]$ with the basis indexed by the elements of a Weyl group W and relations deforming those of $\mathbb{Z}[W]$. The transition matrix from the standard basis to a certain basis (called Kazhdan-Lusztig's basis) is uni-triangular with non-diagonal entries in $v\,\mathbb{Z}_{\geq 0}[v]$. The matrix coefficients evaluated at $v = 1$ give the multiplicities of simple objects in standard ones in the principal block $\mathcal{O}_0$ of category $\mathcal{O}$. The precise formulations are given in Theorem 2.3.

The first proof was provided independently by Beilinston-Bernstein in [BB81] and Brylinski-Kashiwara in [BK81], using the machinery of D-modules and perverse sheaves in the beginning of 1980-s. A decade later Soergel in [Soe90] and [Soe92] suggested a different approach via bimodules over the polynomial ring $R = \mathbb{R}[\mathfrak{h}]$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The independent proof using Soergel's ideas was completed recently by Elias and Williamson in [EW].

The structure of the notes is as follows. In Section 2 we recall the generalities on Hecke algebras associated with finite Weyl groups, introduce the Kazhdan-Lusztig basis and verify its existence and uniqueness. The pivotal point of this section is the statement of Theorem 2.3 (known as the Kazhdan-Lusztig conjecture).

Soergel's approach to the conjecture starts to unravel in Section 3, culminating in Soergel's categorification theorem. In Section 5 we explain the connection of Bott-Samelson modules and bimodules to cohomology and equivariant cohomology of Bott-Samelson varieties.

2. Hecke Algebras

Definition 2.1. Let (W, S) be a Weyl group. The Hecke algebra $\mathcal{H}$ is the algebra over the ring $\mathbb{Z}[v^{\pm 1}]$ with the generators given by the symbols $\{H_s | s \in S\}$ and relations

(2.1)
$$\begin{cases} H_s^2 = (v^{-1} - v)H_s + 1 \Leftrightarrow (H_s + v)(H_s - v^{-1}) = 0 \ \forall s \in S \qquad \text{(quadratic relations)} \\ \underbrace{H_t H_s H_t \dots}_{m_{st}} = \underbrace{H_s H_t H_s \dots}_{m_{st}} \ \forall s, t \in S \qquad \text{(braid relations).} \end{cases}$$

For any element $x \in W$ and a reduced expression $x = s_{i_1} \dots s_{i_k}$, define $H_x := H_{s_{i_1}} \dots H_{s_{i_k}}$. We set H_e to be the unit.

Remark 2.1. As any two reduced expressions of an element $x \in W$ can be obtained from one another by a sequence of braid moves, the element H_x does not depend on the choice of a reduced expression of x.

Remark 2.2. The elements $\langle H_x \rangle_{x \in W}$ generate $\mathcal{H}$ as $\mathbb{Z}[v^{\pm 1}]$-module. One can show that they form a basis.

Exercise 2.1. Check that $H_s^{-1} = H_s + v - v^{-1}$. Therefore, H_x is invertible for any $x \in W$.

There is a ring involution τ on $\mathcal{H}$, given by $\tau : v \mapsto v^{-1}$ and $\tau : H_x \mapsto \overline{H_x} := H_{x^{-1}}^{-1}$.

Definition 2.2. Let w_1, w_2 be in W. Then $w_1 \prec w_2$ in the *Bruhat order* if $w_2 = s_{\beta i_k} \dots s_{\beta_{i_1}} w_1$ and $\ell(s_{\beta_{i_{k-j}}} \dots s_{\beta_{i_1}} w_1) > \ell(s_{\beta_{i_{k-j-1}}} \dots s_{\beta_{i_1}} w_1)$ for all $j \in \{1, \dots, k-1\}$, where $s_{\beta i_j}$ are some (not necessarily simple) reflections in W.

Proposition 2.1. *There exists a basis (the Kazhdan-Lusztig basis)* $\langle b_x \rangle_{x \in W}$ *of* $\mathcal{H}$ *uniquely characterized by two properties:*

$$\tau(b_x) = b_x;$$

(2.2)
$$b_x = H_x + \sum_{y \in W, y \prec x} c_{x,y} H_y,$$

where each $c_{x,y} \in v\mathbb{Z}[v]$.

Remark 2.3. As the transition matrix from $\{H_x\}_{x\in W}$ to $\{b_x\}_{x\in W}$ is upper-triangular with 1's on the diagonal, the elements $\{b_x\}_{x\in W}$, indeed, form a basis of $\mathcal{H}$.

Definition 2.3. The polynomials $p_{y,x} := v^{\ell(x)-\ell(y)}c_{x,y}$ are called the *Kazhdan-Luzstig polynomials.*

Exercise 2.2. The elements $b_s := \{H_s + v\}_{s\in S}$ are self-dual with respect to τ. Check that $b_s^2 = (v + v^{-1})b_s$.

Now we present the proof of Proposition 2.1.

Proof. We first show the existence of a basis, satisfying the required properties, arguing by induction on the Bruhat order. Thus, we set $b_e := 1$ and $b_s := H_s + v$ for $s \in S$ (these are self-dual due to Exercise 2.2) and suppose that b_w exist for $w \prec x$. It is direct to verify that

$$b_sH_x = \begin{cases} H_{sx} + vH_x, \ell(x) < \ell(sx) \\ H_{sx} + v^{-1}H_x, \ell(x) > \ell(sx). \end{cases} \tag{2.3}$$

Next, to find b_x, we use that there exists $s \in S$, such that $sx \prec x$. Hence, using the assumption that b_{sx} exists and formulas (2.3), one can conclude that $b_sb_{sx} = H_x + \sum\limits_{y\prec x} h_yH_y$ for some $h_y \in \mathbb{Z}[v]$ (the containment follows from the existence of $b_{sx} = H_{sx} + \sum\limits_{z\prec sx} h_zH_z$ with $h(z) \in v\,\mathbb{Z}[v^{\pm 1}]$, formulas (2.3) show that the degrees of monomials in h_y are at most one less than the degrees of monomials in polynomials h_z it is derived from), i.e. some of the h_y's might have constant terms. However, subtracting $\sum\limits_{y\prec x} h_y(0)b_y$, we obtain the element b_x, which is fixed by τ (as a $\mathbb{Z}$-linear combination of fixed elements), whose coefficients are polynomials in $v\mathbb{Z}[v]$.

Now we show that b_x is unique. Indeed, if we have two elements $c = H_x+\ldots$ and $c' = H_x+\ldots$, both satisfying (2.2), then $c - c'$ is also stable under τ and $c - c' \in \sum\limits_{y\in W, y\prec x} v\mathbb{Z}[v]H_y$. Now the result follows from Lemma 2.2 below. □

Lemma 2.2. *If* $h \in \sum\limits_{y\in W} v\mathbb{Z}[v]H_y$ *and* $\tau(h) = h$, *then* $h = 0$.

Proof. Let z be one of the maximal elements (in the Bruhat order) in the expression of h in the lemma, i.e. we can write

$$h = p_zH_z + \sum_{y\not\succeq z} p_yH_y,$$

for some polynomials p_z and p_y's in $v\mathbb{Z}[v]$. Now $H_z \in b_z + \sum\limits_{f\prec z} \mathbb{Z}[v^{\pm 1}]b_f$ (for some τ-invariant b_f's, the existence of which was already established). Hence, $\tau(H_z) \in b_z + \sum\limits_{f\prec z} \mathbb{Z}[v^{\pm 1}]b_f \subset H_z + \sum\limits_{f\prec z} \mathbb{Z}[v^{\pm 1}]H_f$. But then $\tau(h) = h$ implies $\tau(p_z) = p_z$, and we obtain a contradiction with the assumption $p_z \in v\mathbb{Z}[v]$. □

Example 2.1. Let us find the Kazhdan-Lusztig basis for the dihedral group $W = \langle s,t\rangle$ with $s^2 = t^2 = e$ and $\underbrace{sts\dots}_{m} = \underbrace{tst\dots}_{m}$. Clearly, $b_e = H_e, b_t = H_t + v$ and $b_s = H_s + v$. Next, $b_sb_t = H_{st} + v(H_s + H_t) + v^2$ satisfies the conditions 2.2, so we put $b_{st} = b_sb_t$, similarly, $b_{ts} = b_tb_s$. Using formulas (2.3), we find $b_sb_{ts} = H_{sts} + v(H_{st} + H_{ts}) + vH_s^2 + v^2(H_t + 2H_s) + v^3$. As $vH_s^2 = v((v^{-1} - v)H_s + 1) = -v^2H_s + H_s + v$, we set $b_{sts} = b_sb_{ts} - b_s = H_{sts} + v(H_{st} + H_{ts}) + vH_s^2 + v^2(H_t + H_s) + v^3$. In general,

$$b_w = H_w + \sum_{x\prec w} v^{\ell(w)-\ell(x)}H_x. \tag{2.4}$$

Indeed, assume that 2.4 holds for $b_{w'}, w' \prec w$. Then, either $sw \prec w$ or $tw \prec w$. Arguing similarly to the proof of Proposition 2.1, one can easily verify the formula for b_w (w.l.o.g. assume $w' = sw \prec w$):

(2.5)
$$b_sb_{w'} = H_w+vH_{w'}+\sum_{x\prec w',x=t\dots} v^{\ell(w')-\ell(x)}H_{sx}+v^{\ell(w')-\ell(x)+1}H_x+\sum_{x\prec w',x=s\dots} v^{\ell(w')-\ell(x)}H_{sx}+v^{\ell(w')-\ell(x)-1}H_x,$$

which is $b_w + b_{tw'}$.

Remark 2.4. In particular, the Weyl groups of types A_2, B_2 and G_2 are dihedral for $m = 3, 4$ and 6. Hence, the Kazhdan-Lusztig basis is given by Example 2.1.

The following result and subsequent remark were conjectured in [KL79] and are proved by now. Theorem 2.3 is known as the Kazhdan-Lusztig conjecture.

Theorem 2.3. *(Kazhdan-Lusztig conjecture) The multiplicity* $[P(x\cdot 0) : \triangle(y\cdot 0)]$ *is given by the specialization of* $c_{x,y}$ *at* $v = 1$ *(using BGG-reciprocity* $[\triangle(y\cdot 0)] : L(x\cdot 0)]$ *equals* $c_{x,y}|_{v=1}$ *as well).*

Remark 2.5.
(1) The polynomials $h_{y,x}$ from 2.2 are in $\mathbb{Z}_{\geq 0}[v]$.
(2) If we write $b_xb_y = \sum \mu^z_{x,y}b_z$, then $\mu^z_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

3. Soergel bimodules

Let (W, S) be a Coxeter system. For any two simple reflections $s, t \in S$, the order of the element $st \in W$ will be denoted by $m_{st} \in \{2, 3, \dots \infty\}$.

Definition 3.1. An *expression* of $w \in W$ is a word $\underline{w} = s_{i_1}\dots s_{i_k}$. The expression $\underline{w}$ is called *reduced* if $\ell(w) = k$.

Next, we fix a vector space $\mathfrak{h}$ over $\mathbb{R}$, s.t. there exist subsets of linearly independent elements $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ with the following properties:

$$\alpha_s(\alpha_t^\vee) = -2\mathrm{Cos}(\frac{\pi}{m_{st}})\ \forall s,t \in S \tag{3.1}$$

$$s\cdot v = v - \alpha_s^\vee(v)\alpha_s\ \forall s \in S, v \in \mathfrak{h}. \tag{3.2}$$

We choose $\mathfrak{h}$ of minimal dimension with the above properties. Let $R = \mathbb{R}[\mathfrak{h}]$ be the coordinate ring of $\mathfrak{h}$. We define the grading on R by setting $\deg(\alpha) = 2$ for any $\alpha \in \mathfrak{h}^*$. In case W is a Weyl group, $\mathfrak{h}_{\mathbb{R}}$ is a real part of the Cartan subalgebra, the α_s's are the roots and $\alpha_t^\vee$'s are the coroots. The augmentation ideal (ideal of nonconstant polynomials) of R will be denoted by R^+.

We consider the abelian category of finitely generated graded R-bimodules. All morphisms preserve the grading (in other words, are homogeneous of degree 0).

Definition 3.2. For any simple reflection $s \in S$ set $B_s := R \otimes_{R^s} R(1)$. We denote by (n) the shift of grading by the corresponding number, i.e. $R \otimes_{R^s} R(1)$ means that the degree of $1 \otimes 1$ is -1, etc. The *Bott-Samelson bimodule* associated to an expression $\underline{w} = s_1 \dots s_m$ is $BS(\underline{w}) = B_{s_1} \otimes_R \dots \otimes_R B_{s_m} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_m}} R(n)$.

By the *Bott-Samelson* module we will understand $BS(\underline{w}) \otimes_R \mathbb{R}$.

Definition 3.3. The operator $R \to R$ given by $\partial_s(r) := \frac{r - s(r)}{2\alpha_s}$ is called the *Demazure operator*. Notice that ∂_s is R^s-linear.

Exercise 3.1. The elements $c_{id} := 1 \otimes 1$ and $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$ (of degrees -1 and 1) form a basis of B_s as a left (or right) R-module. One has relations

$$c_s r = r c_s \tag{3.3}$$

$$r c_{id} = c_{id} s(r) + \partial_s(r) c_s, \tag{3.4}$$

Remark 3.1. In general, one can check, that the elements $c_{\underline{\epsilon}} := c_{\epsilon_{i_1}} \otimes \dots \otimes c_{\epsilon_{i_k}}$, where $\underline{\epsilon} = s_{i_1} \dots s_{i_k}$ runs through all subexpressions of $\underline{w}$ form a basis of $BS(\underline{w})$ as a left (or right) R-module.

Notation 3.1. Henceforth we abbreviate

$$B_{s_{i_1}} \dots B_{s_{i_k}} := B_{s_{i_1}} \otimes_R \dots \otimes_R B_{s_{i_k}}$$

We provide an example of an easy calculation of the product of two Bott-Samelson bimodules.

Example 3.1. Using, $R = R^s \oplus R^s \alpha_s = R^s \oplus R^s(-2)$ (the equality of B_s-bimodules), we write

$$B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R = R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R =$$
$$= B_s(1) \oplus B_s(-1),$$

which is analogous to the relation

$$b_s^2 = (v + v^{-1}) b_s$$

in $\mathcal{H}$ (see Exercise 2.2).

Lemma 3.1. *In Example 5.1 ($W = A_2$), the Bott-Samelson bimodule $BS(\underline{s_1 s_2 s_1})$ decomposes into the direct sum $B_{s_1 s_2 s_1} \oplus B_{s_1}$, where $B_{s_1 s_2 s_1} = R \otimes_{R^W} R(3)$ is the submodule generated by $1 \otimes 1 \otimes 1$.*

Proof. Let us verify this decomposition. The main ingredient of the proof is to produce a nontrivial idempotent of degree 0 in $\mathrm{End}(BS(\underline{s_1 s_2 s_1}))$. For this we define some morphisms between bimodules:

$$m_s \in \mathrm{Hom}(B_s, R) : p \otimes q \mapsto pq$$

$$m_s^a \in \mathrm{Hom}(R, B_s) : 1 \mapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s$$
$$j_s \in \mathrm{Hom}(B_sB_s, B_s) : p \otimes h \otimes q \mapsto p\partial_s(h) \otimes q$$
$$j_s^a \in \mathrm{Hom}(B_s, B_sB_s) : p \otimes q \mapsto p \otimes 1 \otimes q.$$

Notice, that the morphisms m_s and m_s^a have degree 1, while the the degree of the morphisms j_s and j_s^a is -1. Next, let us introduce $e := -m_{s_2}^a j_{s_1}^a j_{s_1} m_{s_2} \in \mathrm{End}(B_{s_1}B_{s_2}B_{s_1}) : B_{s_1}B_{s_2}B_{s_1} \overset{m_{s_2}}{\to} B_{s_1}B_{s_1} \overset{j_{s_1}}{\to} B_{s_1} \overset{j_{s_1}^a}{\to} B_{s_1}B_{s_1} \overset{m_{s_2}^a}{\to} B_{s_1}B_{s_2}B_{s_1}$ and claim that e is an idempotent. Indeed, this follows from the equality $j_{s_1} m_{s_2} m_{s_2}^a j_{s_1}^a \in \mathrm{Hom}(B_{s_1}, B_{s_1}) : B_{s_1} \overset{j_{s_1}^a}{\to} B_{s_1}B_{s_1} \overset{m_{s_2}^a}{\to} B_{s_1}B_{s_2}B_{s_1} \overset{m_{s_2}}{\to} B_{s_1}B_{s_1} \overset{j_{s_1}}{\to} B_{s_1} = -id$ shown below:

$$p \otimes q \overset{j_{s_1}^a}{\mapsto} p \otimes 1 \otimes q \overset{m_{s_2}^a}{\mapsto} p \otimes (\alpha_{s_2} \otimes 1 + 1 \otimes \alpha_{s_2}) \otimes q \overset{m_{s_2}}{\mapsto} 2(p \otimes \alpha_{s_2} \otimes q) \overset{j_{s_1}}{\mapsto} -p \otimes q.$$

The last transition follows from the equality $s_1(\alpha_{s_2}) = \alpha_{s_2} + \alpha_{s_1}$ and Definition 3.3. Hence, e is a projector. As the first two maps in the definition of e are surjective and the last - injective and the chain of maps is $B_{s_1}B_{s_2}B_{s_1} \to B_{s_1}B_{s_1} \to B_{s_1} \to B_{s_1}B_{s_1} \to B_{s_1}B_{s_2}B_{s_1}$, we see that e is the projector onto B_{s_1}. Next, the morphism $1-e$ is a projection as well, so, $B_{s_1}B_{s_2}B_{s_1} = im(e) \oplus im(1-e)$. We first show that $B_{s_1}B_{s_2}B_{s_1}$ is generates by two elements $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes x_1 \otimes 1 \otimes 1$, where $R = \mathbb{R}[x_1, x_2, x_3, x_4]$. Indeed, as $(x_1+x_2)(1 \otimes 1 \otimes 1 \otimes 1) = 1 \otimes (x_1 + x_2) \otimes 1 \otimes 1$ (as $(x_1 + x_2)$ is invariant under s_1), we have $1 \otimes x_2 \otimes 1 \otimes 1$ and, thus $1 \otimes (x_1 - x_2) \otimes 1 \otimes 1 = 1 \otimes \alpha_{s_1} \otimes 1 \otimes 1$ is in the submodule, generated by $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes x_1 \otimes 1 \otimes 1$. Next, $1 \otimes x_1 \otimes 1 \otimes 1 = 1 \otimes 1 \otimes x_1 \otimes 1$ and $1 \otimes 1 \otimes x_3 \otimes 1 = (1 \otimes 1 \otimes 1 \otimes 1)x_3$ and $1 \otimes 1 \otimes (x_1 - x_2) \otimes 1$, thus, $1 \otimes 1 \otimes (x_2 - x_3) \otimes 1 = 1 \otimes 1 \otimes \alpha_{s_2} \otimes 1$ are in the submodule as well. Similarly can be shown that the submodule contains $1 \otimes \alpha_{s_1} \otimes \alpha_{s_2} \otimes 1$ and, therefore, by Remark 3.1 generates the module.

The calculations above, in particular, show that $\dim(\mathbb{R} \otimes_R B_{s_1}B_{s_2}B_{s_1} \otimes_R \mathbb{R}) = 2$. The fact that $\dim(\mathbb{R} \otimes_R B_{s_1}B_{s_2}B_{s_1} \otimes_R \mathbb{R}) = 2$ implies that there are only two indecomposable summands in the decomposition of $BS_{\underline{s_1s_2s_1}}$.

Now define a map of R-bimodules

$$\gamma : R \otimes_{R^W} R(3) \to BS_{\underline{s_1s_2s_1}}$$

by

$$p \otimes q \mapsto p \otimes 1 \otimes 1 \otimes q.$$

Since as left R-modules $BS_{\underline{s_1s_2s_1}} \cong R(-3) \oplus R(-1)^{\oplus 3} \oplus R(1)^{\oplus 3} \oplus R(3)$ (see Remark 3.1) and $R \otimes_{R^W} R(3) \cong R(-3) \oplus R(-1)^{\oplus 2} \oplus R(1)^{\oplus 2} \oplus R(3)$, $im(1-e)$ and $R \otimes_{R^W} R(3)$ have the same graded dimensions as vector spaces over $\mathbb{R}$, it suffices to show that the map γ is surjective. As $(1-e)(1 \otimes 1 \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1$ (follows from $\partial_s(1) = 0$), and $1 \otimes 1 \otimes 1 \otimes 1 \in im(\gamma)$ as well, it suffices to show that $im(1-e)$ is generated by $1 \otimes 1 \otimes 1 \otimes 1$. For this we need to show that the submodule, generated by $1 \otimes 1 \otimes 1 \otimes 1$ contains $im(1-e)(1 \otimes x_1 \otimes 1 \otimes 1)$ First we compute $-e(1 \otimes x_1 \otimes 1 \otimes 1)$:

$$1 \otimes x_1 \otimes 1 \otimes 1 \overset{m_{s_2}}{\mapsto} 1 \otimes x_1 \otimes 1 \overset{j_{s_1}}{\mapsto} \frac{1}{2}(1 \otimes 1) \overset{j_{s_1}^a}{\mapsto} \frac{1}{2}(1 \otimes 1 \otimes 1) \overset{m_{s_2}^a}{\mapsto} \frac{1}{2}(1 \otimes (x_2 - x_3) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (x_2 - x_3) \otimes 1).$$

So, $(1-e)(1\otimes x_1\otimes 1\otimes 1) = 1\otimes x_1\otimes 1\otimes 1+\frac{1}{2}(1\otimes(x_2-x_3)\otimes 1\otimes 1+1\otimes 1\otimes(x_2-x_3)\otimes 1)$. Now we show that this element lies in the submodule generated by $1\otimes 1\otimes 1\otimes 1$. For this we write $1\otimes x_1\otimes 1\otimes 1 = \frac{1}{2}(1\otimes x_1\otimes 1\otimes 1+1\otimes 1\otimes x_1\otimes 1)$ and show that $\frac{1}{2}(1\otimes x_1\otimes 1\otimes 1+1\otimes(x_2-x_3)\otimes 1\otimes 1)$ is in the submodule (for $\frac{1}{2}(1\otimes 1\otimes x_1\otimes 1+1\otimes 1\otimes(x_2-x_3)\otimes 1)$ the computation is completely analogous):

$$\frac{1}{2}(1\otimes x_1\otimes 1\otimes 1+1\otimes(x_2-x_3)\otimes 1\otimes 1) = \frac{1}{2}(1\otimes(x_1+x_2-x_3)\otimes 1\otimes 1 = \frac{1}{2}(x_1+x_2-x_3)\otimes 1\otimes 1\otimes 1.$$

This concludes the proof. □

One should notice the resemblance between the decomposition $B_{s_1}B_{s_2}B_{s_1} = B_{s_1s_2s_1}\oplus B_{s_1}$ and the relation $b_{s_1}b_{s_2}b_{s_1} = b_{s_1s_2s_1}+b_{s_1}$ derived in Example 2.1.

Remark 3.2. More generally, it can be shown that if W is a dihedral group generated by simple reflections (s,t) and $\ell(w')<\ell(w)$, where $w'=sw$, then $B_sB_{w'} = B_w\oplus B_{tw'}$ (compare to (2.5)).

Definition 3.4. The category of *Soergel bimodules* $\mathcal{S}Bim$ is the full subcategory of $\mathbb{Z}$-graded R-bimodules, where the objects are the direct sums of direct summands of graded shifts of BS-bimodules. The morphisms are grading preserving morphisms of $R-R$-bimodules.

Similarly, we define the category of *Soergel modules* $\mathcal{S}Mod$ to be the full subcategory of $\mathbb{Z}$-graded left R-modules, where the objects are the direct sums of direct summands of graded shifts of BS- modules. The morphisms are grading preserving morphisms of R-bimodules.

Remark 3.3. Notice that $BS_{\underline{w_1}}BS_{\underline{w_2}} = BS_{\underline{w_1w_2}}$ implies that the category $\mathcal{S}Bim$ is closed w.r.t the tensor product. As $fg_1\otimes_{R^{s_{i_1}}}g_2\otimes_{R^{s_{i_2}}}\ldots\otimes_{R^{s_{i_n}}}g_n = g_1\otimes_{R^{s_{i_1}}}g_2\otimes_{R^{s_{i_2}}}\ldots\otimes_{R^{s_{i_n}}}g_nf$ for $f\in R^W$, every Soergel bimodule is actually an $R\otimes_{R^W}R$-module.

Definition 3.5. An additive category is said to be *Krull-Schmidt* if every object is isomorphic to a direct sum of indecomposable objects and such decomposition is unique up to isomorphism and permutation of summands.

Proposition 3.2. *The category of Soergel bimodules is Krull-Schmidt.*

Proof. We notice that the category $\mathcal{S}Bim$ is closed under taking direct summands (by its definition). Since the bimodule $\mathrm{Hom}_{R\otimes R}(M,N)$ between any two finitely generated graded bimodules M and N is graded and finitely generated, the degree 0 part is a finite-dimensional space. Thus, the additive category $\mathcal{S}Bim$ is closed under taking direct summands and has finite-dimensional Hom-spaces. It is a standard fact that such categories are Krull-Schmidt. □

Next we explain what we mean by the split Grothendieck group $K_0(\mathcal{S}Bim)$ of the category $\mathcal{S}Bim$. This is the abelian group generated by symbols $[B]$ for all objects $B\in\mathcal{S}Bim$ subject to the relations $[B]=[B']+[B'']$ whenever $B\cong B'\oplus B''$ in $\mathcal{S}Bim$. We make $K_0(\mathcal{S}Bim)$ into a $\mathbb{Z}[v^{\pm 1}]$-module via $v^i[M]=[M](i)$ and $[M]\in K_0(\mathcal{S}Bim)$. The tensor product on $\mathcal{S}Bim$ endows $K_0(\mathcal{S}Bim)$ with multiplication, thus, making it a $\mathbb{Z}[v^{\pm 1}]$-algebra. Moreover, $K_0(\mathcal{S}Bim)$ is a free $\mathbb{Z}[v^{\pm 1}]$ - module, whose basis consists of indecomposable objects (we take one up to a grading shift).

We can now formulate the main theorem.

Theorem 3.3. *(Soergel's categorification theorem) There is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras $\mathcal{H} \to K_0(\mathcal{S}Bim)$, sending b_s to $[B_s]$.*

Corollary 3.4. *(Weak form of Soergel's categorification theorem). There exists a unique homomorphism of rings $c : \mathcal{H} \to K_0(\mathcal{S}Bim)$, s.t. $c(b_s) = B_s$.*

Proof. The quadratic relation was checked in Example 3.1 and the braid relations - in Lemma 3.1 (for the simply laced case) and stated in Remark 3.2 for the general case. The uniqueness of c is obvious, since it is defined on a generating set. □

4. Soergel's categorification theorem

Next we would like to present the classification of indecomposable Soergel bimodules and give a prove of the main theorem. We will use the following proposition (see Section 4 of [Soe92]).

Proposition 4.1. *For two Soergel bimodules B_1, B_2, the canonical map $G : \mathrm{Hom}_{R\otimes R}(B_1, B_2)\otimes_R \mathbb{R} \to \mathrm{Hom}_R(B_1 \otimes_R \mathbb{R}, B_2 \otimes_R \mathbb{R})$ is an isomorphism.*

Corollary 4.2. *The map $\delta : M \mapsto M \otimes_R \mathbb{R}$ induces an embedding of indecomposable objects in $\mathcal{S}Bim$ into indecomposable objects in $\mathcal{S}Mod$.*

Proof. We first show that the image of an indecomposable module M is indecomposable. Indeed, if $\delta(M)$ would decompose as $M_1 \oplus M_2$ there would be a degree zero idempotent $e_{M_1} \in \mathrm{End}_R(M \otimes_R \mathbb{R})$, but since $\mathrm{End}_R(\delta(M)) \cong \mathrm{End}_{R\otimes R}(M) \otimes_R \mathbb{R}$, this implies the existence of a degree zero idempotent (there exists a lift - this is a standard fact, which can be shown by constructing the lifts modulo $(R_+)^n$ for every $n \in \mathbb{N}$ and $(R_+)^n End_{R\otimes R}(M)$ has no degree 0 elements for n large enough) $\widetilde{e} \in \mathrm{End}_{R\otimes R}(M) \otimes_R \mathbb{R}$, which (as $M \cong \widetilde{e}M \oplus (1 - \widetilde{e})M$ and $\widetilde{e} \neq 1$) contradicts our assumption that M is indecomposable.

Next we check that δ maps non isomorphic indecomposables to non isomorphic ones. Assume the contrary and let $M_1, M_2 \in \mathcal{S}Bim$ be indecomposable and $\delta(M_1) \cong \delta(M_2) = \widetilde{M} \in \mathcal{S}Mod$. Then there exist a $\alpha \in \mathrm{Hom}_{R\otimes R}(M_1, M_2)$ and $\beta \in \mathrm{Hom}_{R\otimes R}(M_2, M_1)$, s.t. $G(\alpha \circ \beta)$ is invertible. The application of graded Nakayama's lemma implies $\alpha \circ \beta$ is invertible:

$$\alpha \circ \beta(M_2) + R_+ M_2 = M_2$$

$$R_+ \frac{M_2}{\alpha \circ \beta(M_2)} = \frac{M_2}{\alpha \circ \beta(M_2)}$$

gives $M_2 = \alpha \circ \beta(M_2)$. So $\alpha \circ \beta$ is surjective, hence invertible. □

Theorem 4.3. *Each Bott-Samelson bimodule $BS(\underline{w})$ contains a unique indecomposable summand B_w which does not appear in $BS(\underline{x})$ for $x \prec w$ depends only on w, but not on the reduced expression.*

Proof. Recall from Dmytro's talk (Corollary 5.9) that, for a reduced $\underline{w}$, the module $BS(\underline{w}) \otimes_R \mathbb{R}$ contains a unique graded indecomposable summand, S_w, that does not appear in $BS(\underline{w}') \otimes_R \mathbb{R}$ for shorter $\underline{w}'$ and that depends only on w. In fact, in Dmytro's talk the claim was proved over $\mathbb{C}$ but one can show it holds over $\mathbb{R}$ as well. So $BS(\underline{w}) \otimes_R \mathbb{R} = S_w \oplus \bigoplus_{w'} S_{w'}(d_i)^{\oplus n_i}$ where the sum is taken over $w' \prec w$. Let $BS(\underline{w}) = B_1 \oplus \ldots \oplus B_k$ be the decomposition into indecomposables. By Corollary 4.2 , there is a unique index i (say i=1 to be definite) such that $B_1 \otimes_R \mathbb{R} = S_w$ and then $B_i \otimes_R \mathbb{R} \cong S_{w'}(d_{w'})$ for $i > 1$. We set $B_w := B_1$. Our claim follows from the induction on the length of $\underline{w}$ and Corollary 4.2.

□

Corollary 4.4. *The indecomposable Soergel bimodules are in bijection with the elements of $W \times \mathbb{Z}$.*

Proof. The result follows from the observation that grading shifts preserve indecomposability. □

The above results allow us to prove the main theorem (Theorem 3.3).

Proof. We choose one reduced expression $\underline{w} = \underline{s_1 \dots s_m}$ for every element $w \in W$, then it follows from Theorem 4.3 that the classes of the corresponding $BS(\underline{w})$'s form a basis of $K_0(\mathcal{S}Bim)$ (each $[BS(\underline{w})]$ contains the indecomposable $[B_w]$ as a summand with coefficient 1 and it is not hard to show by induction on the Bruhat order that there exists a $\mathbb{Z}[v^{\pm 1}]$-linear combination of $[BS(\underline{w'})], w' \prec w$ that, being subtracted from $[BS(\underline{w})]$, gives $[B_w]$). Then the corresponding elements $b_{s_1} \dots b_{s_m} \in \mathcal{H}$ are also a basis (again, using induction on the Bruhat order analogously to the proof of Proposition 2.1, we show that b_w is $b_{s_1} \dots b_{s_m}$ minus a $\mathbb{Z}[v^{\pm 1}]$-linear combination of $b_{s_{j_1}} \dots b_{s_{j_k}}$ for $w' = s_{j_1} \dots s_{j_k} \prec w$). □

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