INTRODUCTION TO DEFORMATION THEORY

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ABSTRACT. We give an introduction to deformation theory with a special focus on the moduli space of semistable sheaves and the Quot-scheme. These are notes to a talk given during Spring 2016 at the graduate seminar on moduli of sheaves on K3 surfaces joint between MIT and NEU.

CONTENTS

1. BASIC IDEAS

The goal of deformation theory is to understand the local structure of various moduli spaces. If you are not convinced of the importance of non reduced schemes, deformation theory will change your mind. For a given scheme M over an algebraically closed field k (this is not strictly necessary) we will study morphisms from $\mathrm{Spec}(A)$ to M for any local artinian algebra A over k.

The first question in this direction is what the tangent space of a given point on some moduli space is. This can be studied with maps from $\mathrm{Spec}(D)$ for $D = k[t]/(t^2)$. This algebra is usually called the *dual numbers*. By X we will denote a projective scheme over k. Recall that the *tangent space* $T_P(X)$ of a rational point $P \in X$ is given by $\mathrm{Hom}(m_P/m_P^2, k)$ where m_P is the maximal ideal in the local ring $\mathcal{O}_{X,P}$. The basis for computing tangent spaces in the moduli setting is the following exercise [Har77][Exercise II.2.8].

Proposition 1.1. *Let $P : \mathrm{Spec}(k) \to X$ be a rational point. Then there is a natural bijection*

$$T_P(X) \cong \{f : \mathrm{Spec}(D) \to X : f_{|\mathrm{Spec}(k)} = P\} =: T.$$

Proof. The question is local and we can assume $X = \mathrm{Spec}(R)$. Then P corresponds to a maximal ideal $m \subset R$ such that $R/m \cong k$. We get

$$\begin{aligned} T &\cong \{f : R \to D : f^{-1}((t)) = m\} \\ &\cong \{f : R/m^2 \to D : f^{-1}((t)) = m\} \\ &\cong \mathrm{Hom}(m/m^2, k) = T_P(X) \end{aligned}$$

For the last equality we use $(t) \cong k$ as k-vector spaces. □

A priori T does not have the structure of a k-vector space. For the purpose of this talk we give it such a structure by the bijection outlined in the proof.

Example 1.2. Let H be an ample divisor on X and let $p \in \mathbb{Z}[t]$. By $M^s_{H,p}(X)$ we denote the moduli space of H-Gieseker stable sheaves with Hilbert polynomial p. Assume that this is a fine moduli space, i.e. there is a natural bijection between morphisms from a finite type scheme Z to $M^s_{H,p}(X)$ and flat families of semistable objects with Hilbert polynomial p over Z. Moreover, we fix a stable sheaf F that is parametrized in this moduli space. Then the universal property and the previous proposition lead to a natural bijection between $T_{[F]}(M^s_{H,p}(X))$ and sheaves $F' \in \mathrm{Coh}(X \times_k D)$ that are flat over D together with a morphism $F' \to F$ that restricts to an isomorphism. Such an F' is called a *first order deformation* of F.

We denote the category of local artinian algebras over k by $\underline{\mathrm{Art}}_k$. Morphisms in this category are homomorphisms of k-algebras such that the inverse image of the maximal ideal is the maximal ideal. For any $A \in \underline{\mathrm{Art}}_k$ we write $X_A = X \times_k A$. Moreover, the unique morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ induces a closed embedding $X \hookrightarrow X_A$ that we will use to restrict sheaves from X_A to X.

Definition 1.3. Let $E \in \mathrm{Coh}(X)$ be a coherent sheaf and $A \in \underline{\mathrm{Art}}_k$ a local artinian algebra over k. A *deformation* of E over A is a coherent sheaf $E' \in \mathrm{Coh}(X_A)$ flat over A together with a morphism of $\mathcal{O}_X$-modules $E' \to E$ that restricts to an isomorphism on X. Two deformations are *equivalent* if they are isomorphic as sheaves over X_A and the isomorphism commutes with the two morphisms to E.

2. First Order Deformations of Sheaves

We have a unique morphism $k \to A$ in $\underline{\mathrm{Art}}_k$. It induces a morphism $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$. By abuse of notation we call the induced morphism $X_A \to X$ for any projective scheme X still π. Recall that for any coherent sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras we write $\mathrm{Coh}(\mathcal{A})$ for the category of coherent sheaves on X with an $\mathcal{A}$-module structure.

Lemma 2.1. *The functor $\pi_* : \mathrm{Coh}(X_D) \to \mathrm{Coh}(\pi_* \mathcal{O}_{X_D})$ is an equivalence of categories. Morever, we have $\pi_* \mathcal{O}_{X_D} = \mathcal{O}_X \otimes_k D = \mathcal{O}_X \oplus t\mathcal{O}_X$.*

Proof. Exercise. □

The following basic lemma is our main tool for dealing with flatness.

Lemma 2.2 ([HL10][Lemma 2.1.3])**.** *Let $S_0 \subset S$ be a closed subscheme defined by a nilpotent ideal sheaf $\mathcal{I}$. A coherent sheaf $E \in \mathrm{Coh}(S)$ is flat over S if and only if $E_{|S_0}$ is flat over S_0 and the natural map $\mathcal{I} \otimes_{\mathcal{O}_S} E \to E$ is injective.*

We are now ready to determine first order deformations of sheaves. We follow the proof from [Har10][Theorem 2.7].

Theorem 2.3. *Let E be a coherent sheaf on X. There is a natural bijection between the set of first order deformation of E up to equivalence and $\mathrm{Ext}^1(E, E)$. If $[E] \in M^s_{H,p}(X)$ and $M^s_{H,p}(X)$ is a fine moduli space, then $T_{[E]}(M^s_{H,p}(X)) \cong \mathrm{Ext}^1(E, E)$.*

Proof. Let E' be a deformation of E over D. Tensoring the exact sequence of D-modules

$$0 \to k \xrightarrow{t\cdot} D \to k \to 0$$

with E' leads to an exact sequence

$$0 \to E \xrightarrow{t\cdot} E' \to E \to 0.$$

Therefore, E' represents an element in $\mathrm{Ext}^1(E, E)$. The other way around assume E' is an arbitrary element in $\mathrm{Ext}^1(E, E)$. Then we need to give E' the structure of an $\mathcal{O}_{X_D}$-module. By Lemma 2.1 this can be done by specifying the action of t. We define this to be the composition of $E' \to E$

with $E \to E'$. This definition makes the restriction of $E' \to E$ become an isomorphism. Flatness of E' follows from Lemma 2.2. The second part of the statement about tangent spaces follows as explained in Example 1.2. $\square$

The theorem is still true even if $M^s_{H,p}(X)$ is not a fine moduli space. In order to prove this we will need to study higher order deformations.

3. Higher Order Deformations of Sheaves

In order to study higher order deformation of $E \in \mathrm{Coh}(X)$ we introduce the deformation functor $D_E : \underline{\mathrm{Art}}_k \to \underline{\mathrm{Set}}$ that maps A to the set of deformations of E over A. Let $\sigma : B \twoheadrightarrow A$ be a surjection in $\underline{\mathrm{Art}}_k$. Then there are two natural questions.

1. What is the image of $D_E(\sigma)$, i.e. which deformations on A can be further lifted to B?
2. What are the fibers of $D_E(\sigma)$, i.e. if a deformation can be lifted what are all its lifts?

Definition 3.1. A short exact sequence $0 \to I \to B \to A \to 0$ with $A, B \in \underline{\mathrm{Art}}_k$ and I an ideal in B is called a *small extension* if $m_A \cdot I = 0$, where m_A is the maximal ideal in A.

Let G be a group and S be a set with a G-action. Recall that S is called a *G-torsor* if there is $s \in S$ such that the action of G on s induces a bijection between G and S. Moreover, for sheaves $E, F \in \mathrm{Coh}(X)$ and $i \in \mathbb{Z}$ we write $\hom(E,F) = \dim \mathrm{Hom}(E,F)$ and $\mathrm{ext}^i(E,F) = \dim \mathrm{Ext}^i(E,F)$.

Theorem 3.2 ([HL10][Section 2.A.6])**.** *Let*

$$0 \to I \to B \xrightarrow{\sigma} A \to 0$$

be a small extension and E be a coherent sheaf on X satisfying $\mathrm{Hom}(E,E) = \mathbb{C}$.

1. *The non trivial fibers of $D_E(\sigma)$ are $\mathrm{Ext}^1(E,E) \otimes_k I$-torsors.*
2. *There is a map $o_\sigma : D_E(A) \to \mathrm{Ext}^2(E,E) \otimes_k I$ such that the image of $D_E(\sigma)$ is given by $o_\sigma^{-1}(0)$.*
3. *The image of o_σ lies in a subspace $\mathrm{Ext}^2_0(E,E) \otimes I$, where $\mathrm{Ext}^2_0(E,E) \subset \mathrm{Ext}^2(E,E)$ is of dimension $\mathrm{ext}^2_0(E,E) = \mathrm{ext}^2(E,E) - h^2(\mathcal{O}_X)$.*

Sketch of the Proof. Since $I^2 = 0$ holds, I has an induced structure of an A-module coming from its B-module structure. Any A-module is also a B-module via the morphism $B \twoheadrightarrow A$. Let E_A be a deformation of E over A. Then we have $I \otimes_B E_A = I \otimes_A E_A$. Tensoring the exact sequence of B-modules $0 \to I \to B \to A \to 0$ with E_A shows

$$\mathrm{Tor}^q_B(E_A, A) = \begin{cases} E_A & , \ q = 0 \\ I \otimes_A E_A & , \ q = 1 \\ 0 & , \text{ otherwise.} \end{cases}$$

By adjunction we have an isomorphism of functors

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{X_A}}(E_A \otimes^L_B A, I \otimes_A E_A) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{X_B}}(E_A, I \otimes_A E_A).$$

This induces a spectral sequence

$$E^2_{p,q} = \mathrm{Ext}^p_{\mathcal{O}_{X_A}}(\mathrm{Tor}^q_B(E_A, A), I \otimes_A E_A) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_{X_B}}(E_A, I \otimes_A E_A)$$

that converges on the third sheet, due to our computation of $\mathrm{Tor}^q_B(E_A, A)$. In particular, the group $\mathrm{Ext}^1_{\mathcal{O}_{X_B}}(E_A, I \otimes_A E_A)$ fits into an exact sequence given by

$$0 \to \mathrm{Ext}^1_{\mathcal{O}_{X_A}}(E_A, I \otimes_A E_A) \to \mathrm{Ext}^1_{\mathcal{O}_{X_B}}(E_A, I \otimes_A E_A) \to$$
$$\to \mathrm{Hom}_{\mathcal{O}_{X_A}}(I \otimes_A E_A, I \otimes_A E_A) \to \mathrm{Ext}^2_{\mathcal{O}_{X_A}}(E_A, I \otimes_A E_A).$$

The first map is the one that simply extends $\mathcal{O}_{X_A}$-modules to $\mathcal{O}_{X_B}$-modules via $B \twoheadrightarrow A$. For the second map assume we have an extension E' give by an element in $\operatorname{Ext}^1_{\mathcal{O}_{X_B}}(E_A, I \otimes_A E_A)$. Then the B-module structure induces an action of I on all these modules which we express in the commutative diagram

$$\begin{array}{ccccccccc} & & I \otimes_B I \otimes_A E_A & \longrightarrow & I \otimes_B E' & \longrightarrow & I \otimes_B E_A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & I \otimes_A E_A & \longrightarrow & E' & \longrightarrow & E_A & \longrightarrow & 0. \end{array}$$

The snake lemma and the isomorphism $I \otimes_B E_A \cong I \otimes_A E_A$ induce a map $I \otimes_A E_A \to I \otimes_A E_A$. Similarly to the proof for first order deformations, we want to give E' a structure of B-modules that restricts to E_A. For that the action of I on E' should be given by the map $I \otimes_A E_A \to E'$. We can do that if and only if there are elements in $\operatorname{Ext}^1_{\mathcal{O}_{X_B}}(E_A, I \otimes_A E_A)$ that map to the identity in $\operatorname{Hom}_{\mathcal{O}_{X_A}}(I \otimes_A E_A, I \otimes_A E_A)$. That in turn can only happen if the image of the identity $o_\sigma(E_A) \in \operatorname{Ext}^2_{\mathcal{O}_{X_A}}(E_A, I \otimes_A E_A)$ vanishes.

One has to use some form of cohomology and base change to show $\operatorname{Ext}^i_{\mathcal{O}_{X_A}}(E_A, I \otimes_A E_A) \cong \operatorname{Ext}^i_{\mathcal{O}_X}(E, E) \otimes_k I$ for all $i \in Z$.

To prove part (iii) one needs to construct a trace map $\operatorname{Tr}^i : \operatorname{Ext}^i(E, E) \to H^i(\mathcal{O}_X)$ and proof that the obstruction lies in the kernel of Tr^2. □

4. Deformations of Quotients

Very similarly to the case of sheaves we can study deformations of quotients $E \twoheadrightarrow F$. That will allow us to analyze the local geometry of Quot-schemes.

Definition 4.1. Let $E \twoheadrightarrow F$ be a quotient of coherent sheaves in X and $A \in \underline{\mathrm{Art}}_k$ a local artinian algebra over k. By E_A we denote the pullback of E from X to X_A. A *deformation* of $E \twoheadrightarrow F$ over A is a quotient $E_A \twoheadrightarrow F'$ of coherent sheaves in X_A where F' is flat over A whose restriction $E \twoheadrightarrow F'_{|X}$ is isomorphic to $E \twoheadrightarrow F$. Two deformations are *equivalent* if their kernels are identical. As before we obtain a deformation functor $D_{[E \twoheadrightarrow F]} : \underline{\mathrm{Art}}_k \to \underline{\mathrm{Set}}$.

The theorem describing this functor is very similar to the case of sheaves. A version in the special case of Hilbert schemes is proven in [Har10][Theorem 6.2], while a more general version for the case of the flag scheme is proven in [HL10][Section 2.A.7].

Theorem 4.2. *Let*

$$0 \to I \to B \xrightarrow{\sigma} A \to 0$$

be a small extension and $E \twoheadrightarrow F$ be a quotient of coherent sheaves on X. Moreover, we denote the kernel of $E \twoheadrightarrow F$ by K.

(1) *The non trivial fibers of $D_{[E \twoheadrightarrow F]}(\sigma)$ are $\operatorname{Hom}(K, F) \otimes_k I$-torsors.*
(2) *There is a map $o_\sigma : D_{[E \twoheadrightarrow F]}(A) \to \operatorname{Ext}^1(K, F) \otimes_k I$ such that the image of $D_{[E \twoheadrightarrow F]}(\sigma)$ is given by $o_\sigma^{-1}(0)$.*

5. The Hilbert Scheme of Points on a Surface

The easiest non trivial example is perhaps the Hilbert scheme of n points on a smooth projective surface X denoted by $X^{[n]}$. The locus of reduced subschemes in $X^{[n]}$ is an open subset of the symmetric product $X^{(n)}$ which is of dimension $2n$. It is well known that $X^{[n]}$ is connected, but the techniques are very different from those in these notes. Without using this fact, the following proof will only show that there is a smooth connected component in $X^{[n]}$ of dimension $2n$.

Theorem 5.1. *Let X be a smooth projective surface over k. Then $X^{[n]}$ is a smooth projective variety of dimension $2n$.*

Proof. Let Z be a subscheme of X of dimension 0 and length n. We need to show that the tangent space $T_{[Z]}(X^{[n]})$ has dimension $2n$. By Theorem 4.2 part (1) and the universal property of the Quot-scheme we get $T_{[Z]}(X^{[n]}) = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$. By applying the functor $\mathrm{Hom}(\cdot, \mathcal{O}_Z)$ to the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we obtain the long exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(\mathcal{O}_Z) = \mathbb{C}^n \stackrel{\cong}{\to} \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) = H^0(\mathcal{O}_Z) = \mathbb{C}^n \stackrel{0}{\to} \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \to \\ &\to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = H^1(\mathcal{O}_Z) = 0 \to \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \to \\ &\to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \mathbb{C}^n \to \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) = H^2(\mathcal{O}_Z) = 0 \to \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{O}_Z) \to 0.\end{aligned}$$

Therefore, we can compute

$$\hom(\mathcal{I}_Z, \mathcal{O}_Z) = \chi(\mathcal{I}_Z, \mathcal{O}_Z) + n = 2n.$$

In order to actually prove the equality $\chi(\mathcal{I}_Z, \mathcal{O}_Z) = n$, let $0 \to P^\bullet \to \mathcal{I}_Z \to 0$ be an arbitrary finite locally free resolution of $\mathcal{I}_Z$. We have both $\sum_i \dim(-1)^i r(P^i) = 1$ and $\chi(P^i, \mathcal{O}_Z) = n \cdot r(P^i)$. Additivity of the Euler characteristic implies $\chi(\mathcal{I}_Z, \mathcal{O}_Z) = n$. □

6. Pro-Representability and Dimension Estimates

The moduli space of stable sheaves is not always a fine moduli space, but only a coarse moduli space. This issue can be handled via the following theorem.

Theorem 6.1 ([HL10][Theorem 4.5.1])**.** *Let E be an H-stable coherent sheaf on X with Hilbert polynomial p and $M = M^s_{H,p}(X)$. Then the two functors D_E and $\mathrm{Hom}(\hat{\mathcal{O}}_{M,[E]}, \cdot)$ are naturally isomorphic, where both functors are going from $\underline{\mathrm{Art}}_k$ to $\underline{\mathrm{Set}}$ and* Hom *is taken in the category of local algebras over k.*

If there is an isomorphism as in the theorem one says that D_E is a *pro-representable* functor. The notion of pro-representability can be thought of as having a fine moduli space locally. However, the Zariski topology is too coarse for this to be precisely true.

In the previous example of Hilbert schemes of points on a surface, it is possible to compute the dimension of the moduli space. In general this is much more difficult. Therefore, a homological criterion for smoothness that does not involve knowing the dimension is very useful. In order to obtain this we will have to use higher order deformations.

Theorem 6.2 ([HL10][Theorem 2.A.11])**.** *For all H-stable sheaves E we have the inequality*

$$\mathrm{ext}^1(E, E) \geq \dim_{[E]} M^s_{H,p}(X) \geq \mathrm{ext}^1(E, E) - \mathrm{ext}^2_0(E, E),$$

where $\dim_{[E]} M^s_{H,p}(X)$ denotes the dimension of the component containing $[E]$. In particular, if $\mathrm{ext}^2(E, E) - h^2(\mathcal{O}_X) = 0$, then $M^s_{H,p}$ is smooth at $[E]$. Moreover, if $dim_{[E]} M^s_{H,p}(X) = \mathrm{ext}^1(E, E) - \mathrm{ext}^2_0(E, E)$, then $M^s_{H,p}(X)$ is a complete intersection variety locally at $[E]$.

Proof. Let $R = \hat{\mathcal{O}}_{M,[E]}$ and $m \subset R$ be the maximal ideal. By the previous theorem and Theorem 3.2, we know that the tangent space at $[E]$ has dimension $d = \dim_k m/m^2 = \mathrm{ext}^1(E, E)$. By standard properties of complete local rings, this implies $R \cong k[[t_1, \ldots, t_d]]/J$ for some ideal J. Let $r = \mathrm{ext}^2_0(E, E)$. All statements in the Theorem will follow if we can find r elements that generate J.

The Artin-Rees Lemma says that there is $K > 0$ such that for all integers $N > K$ we have

$$J \cap m^N = m^{N-K}(J \cap m^K) \subset mJ.$$

There is a small extension

$$0 \to I = \frac{J + m^N}{mJ + m^N} \to B = \frac{k[[t_1, \dots, t_d]]}{mJ + m^N} \xrightarrow{\sigma} A = \frac{k[[t_1, \dots, t_d]]}{J + m^N} \to 0.$$

Theorem 6.1 connects the quotient map $R \twoheadrightarrow A = R/m^N$ to a deformation E_A of E. The sheaf E_A can be lifted to B if and only if

$$o_\sigma(E_A) = \sum_{i=1}^{r} \psi_i \otimes \overline{f_i} \in \mathrm{Ext}^2_0(E, E) \otimes_k I$$

vanishes, where $\psi_1, \dots, \psi_r$ is a basis of $\mathrm{Ext}^2_0(E, E)$ and $f_1, \dots, f_r \in J$ are lifts of $\overline{f}_1, \dots, \overline{f}_r \in I$. We get another small extension of the form $\sigma' : C = B/(\overline{f}_1, \dots, \overline{f}_r) \twoheadrightarrow A$. The obstruction

$$o_{\sigma'}(E_A) = \sum_{i=1}^{r} \psi_i \otimes \overline{f_i} \in \mathrm{Ext}^2_0(E, E) \otimes_k I/(\overline{f}_1, \dots, \overline{f}_r)$$

vanishes by definition. Therefore, we can lift E_A to E_C and using pro-representability again we get a commutative diagram

$$\begin{array}{ccc} k[[t_1, \dots, t_d]]/J & \longrightarrow & k[[t_1, \dots, t_d]]/(J + m^N) \\ \downarrow & & \| \\ k[[t_1, \dots, t_d]]/(mJ + (f_1, \dots, f_r) + m^n) & \longrightarrow & k[[t_1, \dots, t_d]]/(J + m^N). \end{array}$$

In particular, we get the inclusions

$$J \subset mJ + (f_1, \dots, f_r) + m^N \subset J + m^N.$$

Recall that $J \cap m^N \subset mJ$ holds and intersect the previous inequality with J to obtain $J = mJ + (f_1, \dots, f_r)$. From standard isomorphism theorems we obtain

$$\begin{aligned}(mJ + (f_1, \dots, f_r) + m^N)/m^N &\cong (J + m^N)/m^N \\ &\cong J/(J \cap m^N) \twoheadrightarrow J/mJ.\end{aligned}$$

In particular, $\overline{f}_1, \dots \overline{f}_r$ generate J/mJ and Nakayama's Lemma allows to conclude that $f_1, \dots, f_r$ generate J. □

7. Moduli of Sheaves on the Projective Plane

We finish the talk by determining smoothness in the special case of $X = \mathbb{P}^2$.

Theorem 7.1. *The moduli space of stable sheaves on $\mathbb{P}^2$ is smooth.*

Proof. Let E be a stable coherent sheaf on $\mathbb{P}^2$. Then Serre duality implies the isomorphism $\mathrm{Ext}^2(E, E) \cong \mathrm{Hom}(E, E(-3))$. This group vanishes because both E and $E(-3)$ are stable, but $\mu(E) > \mu(E(-3))$. Theorem 6.2 finishes the proof. □

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